Errata

The following corrections and other changes have been made at http://dlmf.nist.gov/, and are pending for the Handbook of Mathematical Functions. The Editors thank the users who have contributed to the accuracy of the DLMF Project by submitting reports of possible errors. For confirmed errors, the Editors have made the corrections listed here.

**Version 1.0.26 (March 15, 2020)**

Changes

- In Equation (19.20.11),
  \[ R_J(0, y, z, p) = \frac{3}{2p\sqrt{z}} \ln \left( \frac{16z}{y} \right) - \frac{3}{p} R_C(z, p) + O(y \ln y), \]

**Section 14.30**

In regard to the definition of the spherical harmonics \(Y_{l,m} \), the domain of the integer \(m \) originally written as \(0 \leq m \leq l \) has been replaced with the more general \( |m| \leq l \). Because of this change, in the sentence just below (14.30.2), “tesseral for \( m < l \) and sectorial for \( m = l \)” has been replaced with “tesseral for \( |m| < l \) and sectorial for \( |m| = l \)”. Furthermore, in (14.30.4), \( m \) has been replaced with \( |m| \).

*Reported by Ching-Li Chai on 2019-10-05*

**Equations (22.9.8), (22.9.9) and (22.9.10)**

22.9.8

\[ s_{1,3}^{(4)} s_{2,3}^{(4)} + s_{3,3} s_{1,3}^{(4)} + s_{3,3} s_{1,3}^{(4)} = \frac{\kappa^2 - 1}{k^2} \]

22.9.9

\[ c_{1,3} c_{2,3} + c_{2,3} c_{3,3} + c_{3,3} c_{1,3} = -\frac{\kappa(\kappa + 2)}{(1 + \kappa)^2} \]

22.9.10

\[ d_{1,3}^{(2)} d_{1,3}^{(2)} + d_{2,3}^{(2)} d_{1,3}^{(2)} + d_{3,3}^{(2)} d_{1,3}^{(2)} = d_{1,3}^{(4)} d_{1,3}^{(4)} + d_{2,3}^{(4)} d_{3,3}^{(4)} + d_{3,3}^{(4)} d_{1,3}^{(4)} = \kappa(\kappa + 2) \]

Originally all the functions \( s_{m,p}^{(4)}, c_{m,p}^{(4)}, d_{m,p}^{(2)} \) and \( d_{m,p}^{(4)} \) in Equations (22.9.8), (22.9.9) and (22.9.10) were written incorrectly with \( p = 2 \). These functions have been corrected so that they are written with \( p = 3 \). In the sentence just below (22.9.10), the expression \( s_{m,p} s_{n,p}^{(4)} \) has been corrected to read \( s_{m,p} s_{n,p}^{(4)} \).

*Reported by Juan Miguel Nieto on 2019-11-07*
Other Changes

- In Subsection 1.9(i), just below (1.9.1), a phrase was added which elaborates that $i^2 = -1$.
- Poor spacing in math was corrected in several chapters.
- In Section 1.13, there were several modifications. In Equation (1.13.4), the determinant form of the two-argument Wronskian

\[
\mathcal{W}\{w_1(z), w_2(z)\} = \det \begin{bmatrix} w_1(z) & w_2(z) \\ w'_1(z) & w'_2(z) \end{bmatrix} = w_1(z)w'_2(z) - w_2(z)w'_1(z)
\]

was added as an equality. In Paragraph Wronskian in §1.13(i), immediately below Equation (1.13.4), a sentence was added indicating that in general the n-argument Wronskian is given by $\mathcal{W}\{w_1(z), \ldots, w_n(z)\} = \det \begin{bmatrix} w_{j-1}(z) \\ w_{k-1}(z) \end{bmatrix}$, where $1 \leq j, k \leq n$. Immediately below Equation (1.13.4), a sentence was added giving the definition of the n-argument Wronskian. It is explained just above (1.13.5) that this equation is often referred to as Abel’s identity. Immediately below Equation (1.13.5), a sentence was added explaining how it generalizes for nth-order differential equations. A reference to Ince (1926, §5.2) was added.
- In Section 3.1, there were several modifications. In Paragraph IEEE Standard in §3.1(i), the description was modified to reflect the most recent IEEE 754-2019 Floating-Point Arithmetic Standard IEEE (2019). In the new standard, single, double and quad floating-point precisions are replaced with new standard names of binary32, binary64 and binary128. Figure 3.1.1 has been expanded to include the binary128 floating-point memory positions and the caption has been updated using the terminology of the 2019 standard. A sentence at the end of Subsection 3.1(ii) has been added referring readers to the IEEE Standards for Interval Arithmetic IEEE (2015, 2018). This was suggested by Nicola Torracca.
- In Equation (35.7.3), originally the matrix in the argument of the Gaussian hypergeometric function of matrix argument $\mathbf{2F1}$ was written with round brackets. This matrix has been rewritten with square brackets to be consistent with the rest of the DLMF.

Version 1.0.24 (September 15, 2019)

Equation (33.14.15)

\[
\int_0^\infty \phi_{m,\ell}(r)\phi_{n,\ell}(r) \, dr = \delta_{m,n}
\]

The definite integral, originally written as $\int_0^\infty \phi_{n,\ell}^2(r) \, dr = 1$, was clarified and rewritten as an orthogonality relation. This follows from (33.14.14) by combining it with Dunkl (2003, Theorem 2.2).

Other Changes

- In Paragraph Steed’s Algorithm in §3.10(iii), a sentence was added to inform the reader of alternatives to Steed’s algorithm, namely the Lentz algorithm (see e.g., Lentz (1976)) and the modified Lentz algorithm (see e.g., Thompson and Barnett (1986)).
- In Subsection 19.11(i), a sentence and unnumbered equation

\[
R_C(\gamma - \delta, \gamma) = \frac{-1}{\sqrt{\delta}} \arctan \left( \frac{\sqrt{\delta} \sin \theta \sin \phi \sin \psi}{\alpha^2 - 1 - \alpha^2 \cos \theta \cos \phi \cos \psi} \right)
\]

were added which indicate that care must be taken with the multivalued functions in (19.11.5). See (Cayley, 1961, pp. 103-106). This was suggested by Albert Groenenboom.
- In Subsection 33.14(iv), there were several modifications. Just below (33.14.9), the constraint described in the text “$\ell < (-\epsilon)^{-1/2}$ when $\epsilon < 0$,” was removed. In Equa-
Errata

Equation (33.14.13), the constraint $\epsilon_1, \epsilon_2 > 0$ was added. In the line immediately below (33.14.13), it was clarified that $s(\epsilon, \ell; r)$ is $\exp(-r/n)$ times a polynomial in $r/n$, instead of simply a polynomial in $r$. In Equation (33.14.14), a second equality was added which relates $\phi_{n,\ell}(r)$ to Laguerre polynomials. A sentence was added immediately below (33.14.15) indicating that the functions $\phi_{n,\ell}$, $n = \ell, \ell + 1, \ldots$, do not form a complete orthonormal system.

Version 1.0.23 (June 15, 2019)

Equation (17.9.3)

$$2 F_1 \left( \frac{a, b}{c}; q, z \right) = \frac{(abz/c; q)_\infty}{(bcz/q; q)_\infty} 3 F_2 \left( a, c/b, 0 \middle| q, q \right) + \frac{(a, b/c; q)_\infty}{(c, z; (bcz/q); q)_\infty} 3 F_2 \left( z, abz/c, 0 \middle| q, q \right)$$

Originally, the second term on the right-hand side was missing. The form of the equation where the second term is missing is correct if the $2 F_1$ is terminating. It is this form which appeared in the first edition of Gasper and Rahman (1990). The more general version which appears now is what is reproduced in Gasper and Rahman (2004, (III.5)).

Reported by Roberto S. Costas-Santos on 2019-04-26

Equation (23.12.2)

$$\zeta(z) = \frac{\pi^2}{4\omega_1^2} \left( \frac{z}{3} + \frac{2\omega_1}{\pi} \cot \left( \frac{\pi z}{2\omega_1} \right) - 8 \left( z - \frac{\omega_1}{\pi} \sin \left( \frac{\pi z}{\omega_1} \right) \right) q^2 + O(q^4) \right)$$

Originally, the factor of 2 was missing from the denominator of the argument of the cot function.

Reported by Blagoje Oblak on 2019-05-27

Other Changes

- In Equations (15.6.1)–(15.6.9), the Olver hypergeometric function $F(a, b; c; z)$, previously omitted from the left-hand sides to make the formulas more concise, has been added. In Equations (15.6.1)–(15.6.5), (15.6.7)–(15.6.9), the constraint $|\text{ph}(1 - z)| < \pi$ has been added. In (15.6.6), the constraint $|\text{ph}(-z)| < \pi$ has been added. In Section 15.6, the sentence immediately following (15.6.9), “These representations are valid when $|\text{ph}(1 - z)| < \pi$, except (15.6.6) which holds for $|\text{ph}(-z)| < \pi$.”, has been removed.
- In Subsection 25.2(ii), it is now mentioned that (25.2.5), defines the Stieltjes constants $\gamma_n$. Consequently, $\gamma_n$ in (25.2.4), (25.6.12) are now identified as the Stieltjes constants.
- In (25.11.36) we have emphasized the link with the Dirichlet $L$-function, and used the fact that $\chi(k) = 0$. A sentence just below (25.11.36) was added indicating that one should make a comparison with (25.15.1) and (25.15.3).
- Additional keywords are being added to formulas (an ongoing project); these are visible in the associated ‘info boxes’ linked to the $\mathcal{Z}$ icons to the right of each formula, and provide better search capabilities.

Version 1.0.22 (March 15, 2019)
Subsection 14.2(iii)

Previously the exponents of the associated Legendre differential equation (14.2.2) at infinity were given incorrectly by \{-ν - 1, ν\}. These were replaced by \{ν + 1, -ν\}.

Reported by Hans Volkmer on 2019-01-30

Subsection 18.15(i)

In the line just below (18.15.4), it was previously stated “is less than twice the first neglected term in absolute value.” It now states “is less than twice the first neglected term in absolute value, in which one has to take \(\cos θ_{n,m,ℓ} = 1\).”

Reported by Gergő Nemes on 2019-02-08

Equation (33.11.1)

\[ H_{±}^k(η, ρ) \sim e^{±iθ_k(η, ρ)} \sum_{k=0}^{∞} \frac{(a)_k(b)_k}{k!(±2ρ)^k} \]

Previously this formula was expressed as an equality. Since this formula expresses an asymptotic expansion, it has been corrected by using instead an \(\sim\) relation.

Reported by Gergő Nemes on 2019-01-29

Other Changes

- Some references were added to §§7.25(ii), 7.25(iii), 7.25(vi), 8.28(ii), and to Paragraph Products in §10.74(vii), and §10.77(ix).

Version 1.0.21 (December 15, 2018)

Equation (10.22.72)

\[ \int_{0}^{∞} J_μ(at)J_ν(bt)J_ν(ct)t^{1-μ} dt = \frac{(bc)^{μ-1} \sin((μ - ν)π)(\sinh χ)^{μ-\frac{1}{2}}}{(\frac{1}{2}π)^{\frac{1}{2}}a^{μ}} e^{(μ-\frac{1}{2})iπ}Q_{ν-\frac{1}{2}}^{\frac{1}{2}-μ}(cosh χ), \]

\(\Re μ > -\frac{1}{2}, \Re ν > -1, a > b + c, \cosh χ = (a^2 - b^2 - c^2)/(2bc)\)

Originally, the factor on the right-hand side was written as \(\frac{(bc)^{μ-1} \sin((ν+μ)π)(\sinh χ)^{μ-\frac{1}{2}}}{(\frac{1}{2}π)^{\frac{1}{2}}a^{μ}}\), which was taken directly from Watson (1944, p. 412, (13.46.5)), who uses a different normalization for the associated Legendre function of the second kind \(Q^\nu_μ\). Watson’s \(Q^\nu_μ\) equals \(\frac{\sin((ν+μ)π)}{\sin(πν)}e^{μπi}Q^\nu_μ\) in the DLMF.

Reported by Arun Ravishankar on 2018-10-22

Subsection 26.7(iv)

In the final line of this subsection, \(W_m(n)\) was replaced by \(W_p(n)\) twice, and the wording was changed from “or, equivalently, \(N = e^{W_m(n)}\)” to “or, specifically, \(N = e^{W_p(n)}\).”

Reported by Gergő Nemes on 2018-04-09
Equations (31.16.2) and (31.16.3)

31.16.2  \( xy = a \sin^2 \theta \cos^2 \phi, \)  \( (x-1)(y-1) = (1-a)\sin^2 \theta \sin^2 \phi, \)  \( (x-a)(y-a) = a(a-1)\cos^2 \theta \)

31.16.3  \[ A_0 = \frac{n!}{(\gamma + \delta)_n} H_{p_{n,m}}(1), \quad Q_0 A_0 + R_0 A_1 = 0 \]

Originally \( x, y \) were incorrectly defined by the set of equations (31.16.2), given previously as “\( x = \sin^2 \theta \cos^2 \phi, \)  \( y = \sin^2 \theta \sin^2 \phi \)”. In fact, \( x, y \) are implicitly defined by the corrected set of equations. In (31.16.3), the initial data \( A_0 \), previously missing, has now been included.

Other Changes

- In (5.11.14), the previous constraint \( \Re(b-a) > 0 \) was removed, see Fields (1966, (3)).
- In Paragraph Confluent Hypergeometric Functions in §7.18(iv), a note about the multivalued nature of the Kummer confluent hypergeometric function of the second kind \( U \) on the right-hand side of (7.18.10) was inserted.
- In regard to (25.14.1), the previous constraint \( a \neq 0, -1, -2, \ldots \) was removed. A clarification regarding the correct constraints for Lerch’s transcendent \( \Phi(z, s, a) \) has been added in the text immediately below. In particular, it is now stated that if \( s \) is not an integer then \( |\phi a| < \pi \); if \( s \) is a positive integer then \( a \neq 0, -1, -2, \ldots \); if \( s \) is a non-positive integer then \( a \) can be any complex number.

Version 1.0.20 (September 15, 2018)

Changes

- The constraint \( a \neq 0 \) was added in (4.8.14).
- In Chapter 18, the reference Ismail (2005) has been replaced throughout by the further corrected paperback version Ismail (2009).
- In Section 36.1, the entry for * to represent complex conjugation was removed (see Version 1.0.19).
- In (36.2.18) and in §§36.12(i), 36.15(i), 36.15(ii), the vector at the origin, previously given as 0, has been clarified to read 0.
- A software bug that had corrupted some figures has been corrected.

Version 1.0.19 (June 22, 2018)

Equation (33.6.5)

\[ H_{\ell}^{\pm}(\eta, \rho) = \frac{e^{\pm i\phi(\eta, \rho)}}{(2\ell + 1)!\Gamma(-\ell \pm i\eta)} \left( \sum_{k=0}^{\infty} \frac{(a)_k}{(2\ell + 2)_k k!} (\mp 2i\rho)^{a+k} \left( \ln(\mp 2i\rho) + \psi(a+k) - \psi(1+k) - \psi(2\ell + 2 + k) \right) \right) - \sum_{k=1}^{2\ell+1} \frac{(2\ell + 1)! (k-1)!}{(2\ell + 1 - k)! (1-a)_k} (\mp 2i\rho)^{a-k} \]

Originally the factor in the denominator on the right-hand side was written incorrectly as \( \Gamma(-\ell \pm i\eta) \). This has been corrected to \( \Gamma(-\ell \pm i\eta) \).

Reported by Ian Thompson on 2018-05-17
**Sections 33.10(ii), 33.10(iii)**

Originally it was stated incorrectly that \( \rho \) was fixed. This has been corrected to state that \( \eta \rho \) is fixed.

*Reported by Ian Thompson on 2018-05-17*

**Equation (33.11.1)**

\[
H^\pm_{\ell}(\eta, \rho) = e^{\pm i \theta_{\ell}(\eta, \rho)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (\pm 2i \rho)^k}
\]

Originally the factor in the denominator on the right-hand side was written incorrectly as \((\mp 2i \rho)^k\). This has been corrected to \((\pm 2i \rho)^k\).

*Reported by Ian Thompson on 2018-05-17*

**Other Changes**

- The overloaded operator \(\equiv\) is now more clearly separated (and linked) to two distinct cases: equivalence by definition (in §§1.4(ii), 1.4(v), 2.7(i), 2.10(iv), 3.1(i), 3.1(iv), 4.18, 9.18(ii), 9.18(vi), 18.2(iv), 20.2(iii), 20.7(vi), 23.20(ii), 25.10(i), 26.15, 31.17(i)); and modular equivalence (in §§24.10(i), 24.10(ii), 24.10(iii), 24.10(iv), 24.15(iii), 24.19(ii), 26.14(i), 26.21, 27.2(i), 27.8, 27.9, 27.11, 27.12, 27.14(vi), 27.17, 27.15, 27.16, 27.19).

- The notation and markup for complex conjugation has been made more consistent in §§1.17(iii), 9.9(i), 10.11, 10.34, 10.63(ii), 12.11(ii), 13.7(i), 14.30(ii), 23.5(iv), 28.12(ii), 31.15(iii), 34.3(vii), 36.2(iii), 36.2(iv), 36.8, 36.11.

- In Chapter 35, the generalized hypergeometric function of matrix argument \(pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; T)\), was labeled inadvertently as its single variable counterpart \(pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; T)\). Furthermore, the Jacobi function of matrix argument \(P^{(\gamma, \delta)}_{\nu}(T)\), and the Laguerre function of matrix argument \(L^{(\gamma)}_{\nu}(T)\), were also labeled inadvertently (and incorrectly) in terms of the single variable counterparts given by \(P^{(\gamma, \delta)}_{\nu}(T)\), and \(L^{(\gamma)}_{\nu}(T)\). In order to resolve these inconsistencies, these functions now link correctly to their respective definitions.

**Version 1.0.18 (March 27, 2018)**

**Table 5.4.1**

The table of extrema for the Euler gamma function \(\Gamma\) had several entries in the \(x_n\) column that were wrong in the last 2 or 3 digits. These have been corrected and 10 extra decimal places have been included.
Other Changes

- The factor on the right-hand side of Equation (10.9.26) containing \(\cos(\mu - \nu)\theta\) has been replaced with \(\cos((\mu - \nu)\theta)\) to clarify the meaning.

- In Paragraph Confluent Hypergeometric Functions in §10.16, several Whittaker confluent hypergeometric functions were incorrectly linked to the definitions of the Kummer confluent hypergeometric and parabolic cylinder functions. However, to the eye, the functions appeared correct. The links were corrected.

- In Equation (15.6.9), it was clarified that \(\lambda \in \mathbb{C}\).

- Originally Equation (19.16.9) had the constraint \(a, a' > 0\). This constraint was replaced with \(b_1 + \cdots + b_n > a > 0\), \(b_j \in \mathbb{R}\). It therefore follows from Equation (19.16.10) that \(a' > 0\). The last sentence of Subsection 19.16(ii) was elaborated to mention that generalizations may also be found in Carlson (1977). These were suggested by Bastien Roucariès.

- In Section 19.25(vi), the Weierstrass lattice roots \(e_j\), were labeled inadvertently as the base of the natural logarithm. In order to resolve this inconsistency, the lattice roots \(e_j\), and lattice invariants \(g_2, g_3\), now link to their respective definitions (see §§23.2(i), 23.3(i)). This was reported by Felix Ospald.

- In Equation (19.25.37), the Weierstrass zeta function was incorrectly linked to the definition of the Riemann zeta function. However, to the eye, the function appeared correct. The link was corrected.

- In Equation (27.12.5), the term originally written as \(\sqrt{\ln x}\) was rewritten as \((\ln x)^{1/2}\) to be consistent with other equations in the same subsection.

Version 1.0.17 (December 22, 2017)

Paragraph Mellin–Barnes Integrals in §8.6(ii)

The descriptions for the paths of integration of the Mellin-Barnes integrals (8.6.10)–(8.6.12) have been updated. The description for (8.6.11) now states that the path of integration is to the right of all poles. Previously it stated incorrectly that the path of integration had to separate the poles of the gamma function from the pole at \(s = a\) for \(\Gamma(a, z)\). The paths of integration for (8.6.10) and (8.6.12) have been clarified. In the case of (8.6.10), it separates the poles of the gamma function from the pole at \(s = a\) for \(\gamma(a, z)\). In the case of (8.6.12), it separates the poles of the gamma function from the poles at \(s = 0, 1, 2, \ldots\).

Reported 2017-07-10 by Kurt Fischer.

Section 10.37

In §10.37, it was originally stated incorrectly that (10.37.1) holds for \(|\text{ph} z| < \pi\). The claim has been updated to \(|\text{ph} z| \leq \frac{\pi}{2}\).
Equation (18.27.6)

\[ P_n^{(\alpha, \beta)}(x; c, d; q) = \frac{e^{\alpha q^{-1}(\alpha + 1)_n(x^2 - q^{\alpha + 1}c^{-1}d; q)_n}}{(q; q)_n} \times P_n(q^{\alpha + 1}x; q^\alpha, q^\beta, -q^\alpha c^{-1}d; q) \]

Originally the first argument to the big $q$-Jacobi polynomial on the right-hand side was written incorrectly as $q^{\alpha + 1}c^{-1}dx$.

Reported 2017-09-27 by Tom Koornwinder.

Equation (21.6.5)

\[ T = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \]

Originally the prefactor $\frac{1}{2}$ on the right-hand side was missing.

Reported 2017-08-12 by Wolfgang Bauhardt.

Equation (27.12.8)

\[ \text{li}(x) = O\left(x \exp\left(-\lambda(\alpha)(\ln x)^{1/2}\right)\right), \]

\[ m \leq (\ln x)^{\alpha}, \quad \alpha > 0 \]

Originally the first term was given incorrectly by $\frac{x}{\phi(m)}$.

Reported 2017-12-04 by Gergő Nemes.

Other Changes

- In §5.2(iii), three new identities for Pochhammer’s symbol (5.2.6)–(5.2.8) have been added at the end of this subsection. This was suggested by Tom Koornwinder.
- Originally named as a complementary error function, (7.2.3) has been renamed as the Faddeeva (or Faddeyeva) function $w(z)$.
- In §7.8, an inequality (7.8.8) was added at the end of this section. This is Pólya (1949, (1.5)) and was suggested by Roberto Iacono.
- Originally the function $\chi$, used in (9.7.3) and (9.7.4), was presented with argument given by a positive integer $n$. It has now been clarified to be valid for argument given by a positive real number $x$.
- Bounds have been sharpened in §9.7(iii). The second paragraph now reads, “The $n$th error term is bounded in magnitude by the first neglected term multiplied by $\chi(n + \sigma) + 1$ where $\sigma = \frac{1}{8}$ for (9.7.7) and $\sigma = 0$ for (9.7.8), provided that $n \geq 0$ in the first case and $n \geq 1$ in the second case.” Previously it read, “In (9.7.7) and (9.7.8) the $n$th error term is bounded in magnitude by the first neglected term multiplied by $2\chi(n) \exp(\sigma \pi/(72\xi))$ where $\sigma = 5$ for (9.7.7) and $\sigma = 7$ for (9.7.8), provided that $n \geq 1$ in both cases.” In Equation (9.7.16)

\[ \text{Bi}(x) \leq \frac{e^{\xi}}{\sqrt{\pi}x^{1/4}} \left(1 + \left(\frac{n}{72\xi}\right) + \frac{5}{72\xi}\right), \]

\[ \text{Bi}'(x) \leq \frac{x^{1/4}e^{\xi}}{\sqrt{\pi}} \left(1 + \left(\frac{n}{72\xi}\right) + \frac{7}{72\xi}\right), \]

the bounds on the right-hand sides have been sharpened. The factors $\chi\left(\frac{5}{72}\right) + 1$, $\left(\frac{n}{72}\right) + 1$, were originally given by $\frac{5\pi}{72\xi} \exp\left(\frac{5\pi}{72\xi}\right)$, $\frac{7\pi}{72\xi} \exp\left(\frac{7\pi}{72\xi}\right)$, respectively.

- Bounds have been sharpened in §9.7(iv). The first paragraph now reads, “The $n$th error term in (9.7.5) and (9.7.6) is bounded in magnitude by the first neglected term multiplied by

\[ \left\{ \begin{array}{ll} 1, & |\text{ph } z| \leq \frac{1}{3} \pi, \\
\min\left[|\csc(\text{ph } \zeta)|, \chi(n + \sigma + 1), \right], & \frac{1}{3} \pi \leq |\text{ph } z| \leq \frac{2}{3} \pi, \\
\frac{\sqrt{2\pi(n + \sigma)}}{\csc(\text{ph } \zeta)} + \chi(n + \sigma) + 1, & \frac{2}{3} \pi \leq |\text{ph } z| < \pi, \end{array} \right. \]

provided that $n \geq 0$, $\sigma = \frac{1}{8}$ for (9.7.5) and $n \geq 1$, $\sigma = 0$ for (9.7.6).” Previously it read, “When $n \geq 1$ the $n$th error term in (9.7.5) and (9.7.6) is bounded in magnitude by the first neglected term multiplied by

\[ \left\{ \begin{array}{ll} 2 \exp\left(\frac{\sigma}{36\xi}\right), & |\text{ph } z| \leq \frac{1}{3} \pi, \\
2\chi(n) \exp\left(\frac{\sigma \pi}{72\xi}\right), & \frac{1}{3} \pi \leq |\text{ph } z| \leq \frac{2}{3} \pi, \\
4\chi(n) \exp\left(\frac{\sigma \pi}{36\xi}\right), & \frac{2}{3} \pi \leq |\text{ph } z| < \pi. \end{array} \right. \]

Here $\sigma = 5$ for (9.7.5) and $\sigma = 7$ for (9.7.6).”

- In §10.8, a sentence was added just below (10.8.3) indicating that it is a rewriting of (16.12.1). This was suggested by Tom Koornwinder.
- Equations (10.15.1), (10.38.1), have been generalized to include the additional cases of $\partial J_{-\nu}(z)/\partial \nu$, $\partial I_{-\nu}(z)/\partial \nu$, respectively.
• The Kronecker delta symbols in Equations (10.22.37), (10.22.38), (14.17.6)–(14.17.9), have been moved furthest to the right, as is common convention for orthogonality relations.
• The titles of §§14.5(ii), 14.5(vi), have been changed to μ = 0, ν = 0, 1, and Addendum to §14.5(ii) : μ = 0, ν = 2, respectively, in order to be more descriptive of their contents.
• The second and the fourth lines of (19.7.2) containing \( k'/ik \) have both been replaced with \(-ik'/k \) to clarify the meaning.
• Originally Equation (25.2.4) had the constraint \( \Re s > 0 \). This constraint was removed because, as stated after (25.2.1), \( \zeta(s) \) is meromorphic with a simple pole at \( s = 1 \), and therefore \( \zeta(s) - (s - 1)^{-1} \) is an entire function. This was suggested by John Harper.
• The title of §32.16 was changed from Physical to Physical Applications.
• Bibliographic citations and clarifications have been added, removed, or modified in §§5.6(i), 5.10, 7.8, 7.25(iii), and 32.16.

Version 1.0.16 (September 18, 2017)

Equation (8.12.18)

\[
\begin{align*}
Q(a, z) = & & \frac{z^{-a\frac{1}{2}}e^{-\frac{z}{2}}}{\Gamma(a)} \left( \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{A_k (\chi)}{z^{k/2}} + \sum_{k=1}^{\infty} B_k (\chi) \right) \\
\end{align*}
\]

The original \pm in front of the second summation was replaced by \mp to correct an error in Paris (2002); for details see https://arxiv.org/abs/1611.00548.


Equation (14.5.14)

\[
Q_{\nu}^{-1/2}(\cos \theta) = \left( \frac{\pi}{2 \sin \theta} \right)^{1/2} \cos \left( \frac{\nu + \frac{1}{2}}{2} \theta \right)
\]

Originally this equation was incorrect because of a minus sign in front of the right-hand side.

Reported 2017-04-10 by André Greiner-Petter.

Equations (17.2.22) and (17.2.23)

\[
\begin{align*}
17.2.22 & \quad \frac{(qa^2, q^{-a}; q)_n}{(a^2, -a^{2}; q)_n} = \frac{1 -aq^{2n}}{1-a} \\
17.2.23 & \quad \frac{(qa^2, q^{-a}; q)_n}{(a^{2k}, q^{-a}; q)_n} = \frac{1 -aq^{2n}}{1-a} \\
\end{align*}
\]

The numerators of the leftmost fractions were corrected to read \( (qa^2, -qa^2; q)_n \) and \( (qa^2, q^{-a}; q)_n \) instead of \( (qa^2, -aq^2; q)_n \) and \( (aq^2, q^{-a}; q)_n \) respectively.

Reported 2017-06-26 by Jason Zhao.

Figure 20.3.1

![Figure 20.3.1](https://example.com/figure20.3.1.png)

Figure 20.3.1 \( \theta_j(x, \theta_0), 0 \leq x \leq 2, j = 1, 2, 3, 4. \)

The locations of the tick marks denoting 1.5 and 2 on the x-axis were corrected.

Reported 2017-05-22 by Paul Abbott.
V_m(\xi) \sim \frac{1}{2\pi h} \left( -D_{m+2}(\xi) - m(m-1)D_{m-2}(\xi) \right) + \frac{1}{210h^2} \left( D_{m+6}(\xi) + (m^2 - 25m - 36)D_{m+2}(\xi) \right. \\
- m(m-1)(m^2 + 27m - 10)D_{m-2}(\xi) - 6! \left( \frac{m}{6} \right) D_{m-6}(\xi) + \cdots

Originally the $-$ in front of the $6!$ was given incorrectly as $+$. Reported 2017-02-02 by Daniel Karlsson.

### Other Changes

- To be consistent with the notation used in (8.12.16), Equation (8.12.5) was changed to read

\[ \frac{e^{\pm \pi a}}{2 \sin(\pi a)} Q(-a, z e^{\pm \pi i}) \]

\[ = \pm e^{T(a, \eta)} \text{erfc}\left( \pm \eta \sqrt{a/2} \right) \]

- Following a suggestion from James McTavish, on 2017-04-06, the recurrence relation \( u_k = \frac{1}{(6k-5)(6k-3)(6k-1)} u_{k-1} \) was added to Equation (9.7.2). In §15.2(ii), the unnumbered equation

\[ \lim_{c \to -n} \frac{F(a, b; c; z)}{\Gamma(c)} = \frac{(a)_n (b)_n (z + 1)_n + 1}{(n + 1)!} \]

\[ \frac{F(a + n + 1, b + n + 1; n + 2; z)}{n = 0, 1, 2, \ldots} \]

was added in the second paragraph. An equation number will be assigned in an expanded numbering scheme that is under current development. Additionally, the discussion following (15.2.6) was expanded.

- In §15.4(i), due to a report by Louis Klauder on 2017-01-01, and in §15.4(iii), sentences were added specifying that some equations in these subsections require special care under certain circumstances. Also, (15.4.6) was expanded by adding the formula \( F(a, b; a; z) = (1 - z)^{-b} \).

- A bibliographic citation was added in §11.13(i).

### Version 1.0.15 (June 1, 2017)

Changes

- There have been extensive changes in the notation used for the integral transforms defined in §1.14. These changes are applied throughout the DLMF. The following table summarizes the changes.

<table>
<thead>
<tr>
<th>Transform</th>
<th>New Notation</th>
<th>Abbreviated Notation</th>
<th>Old Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fourier</td>
<td>( \mathcal{F}(f)(x) )</td>
<td>( \mathcal{F} f(x) )</td>
<td>( \mathcal{F} f(x) )</td>
</tr>
<tr>
<td>Fourier Cosine</td>
<td>( \mathcal{F}_{c}(f)(x) )</td>
<td>( \mathcal{F}_{c} f(x) )</td>
<td>( \mathcal{F}_{c} f(x) )</td>
</tr>
<tr>
<td>Fourier Sine</td>
<td>( \mathcal{F}_{s}(f)(x) )</td>
<td>( \mathcal{F}_{s} f(x) )</td>
<td>( \mathcal{F}_{s} f(x) )</td>
</tr>
<tr>
<td>Laplace</td>
<td>( \mathcal{L}(f)(s) )</td>
<td>( \mathcal{L} f(s) )</td>
<td>( \mathcal{L}(f(t); s) )</td>
</tr>
<tr>
<td>Mellin</td>
<td>( \mathcal{M}(f)(s) )</td>
<td>( \mathcal{M} f(s) )</td>
<td>( \mathcal{M}(f; s) )</td>
</tr>
<tr>
<td>Hilbert</td>
<td>( \mathcal{H}(f)(s) )</td>
<td>( \mathcal{H} f(s) )</td>
<td>( \mathcal{H}(f; s) )</td>
</tr>
<tr>
<td>Stieltjes</td>
<td>( S(f)(s) )</td>
<td>( S f(s) )</td>
<td>( S(f; s) )</td>
</tr>
</tbody>
</table>

Previously, for the Fourier, Fourier cosine and Fourier sine transforms, either temporary local notations were used or the Fourier integrals were written out explicitly.

- Several changes have been made in §1.16(vii) to

(i) make consistent use of the Fourier transform notations \( \mathcal{F}(f) \), \( \mathcal{F}(\phi) \) and \( \mathcal{F}(u) \) where \( f \) is a function of one real variable, \( \phi \) is a test function of \( n \) variables associated with tempered distributions, and \( u \) is a tempered distribution (see (1.14.1), (1.16.29) and (1.16.35));

(ii) introduce the partial differential operator \( D \) in (1.16.30);

(iii) clarify the definition (1.16.32) of the partial differential operator \( P(D) \); and

(iv) clarify the use of \( P(D) \) and \( P(x) \) in (1.16.33), (1.16.34), (1.16.36) and (1.16.37).

- An entire new Subsection 1.16(viii) Fourier Transforms of Special Distributions, was contributed by Roderick Wong.

- The validity constraint \(| \text{ph} z | < \frac{1}{2} \pi \) was added to (9.5.6). Additionally, specific source citations are now given in the metadata for all equations in Chapter 9.
• The relation between Clebsch-Gordan and $3j$ symbols was clarified, and the sign of $m_3$ was changed for readability. The reference Condon and Shortley (1935) for the Clebsch-Gordan coefficients was replaced by Edmonds (1974) and Rotenberg et al. (1959) and the references for $3j$, $6j$, $9j$ symbols were made more precise in §34.1.

• The website’s icons and graphical decorations were upgraded to use SVG, and additional icons and mouse-cursors were employed to improve usability of the interactive figures.

**Version 1.0.14 (December 21, 2016)**

**Equation (8.18.3)**

\[ I_x(a, b) = \frac{\Gamma(a + b)}{\Gamma(a)} \left( \sum_{k=0}^{n-1} d_k F_k + O(a^{-n}) F_0 \right) \]

The range of $x$ was extended to include 1. Previously this equation appeared without the order estimate as $I_x(a, b) \sim \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{k=0}^{\infty} d_k F_k$.

*Reported 2016-08-30 by Xinrong Ma.*

**Equation (17.9.2)**

\[ _2\phi_1 \left( \frac{q^{-n}, b}{c}; q, z \right) = \frac{(c/b; q)_n}{(c; q)_n} b^n 3\phi_1 \left( \frac{q^{-n}, b, q/z}{bq^{1-n}/c}; q, z/c \right) \]

The entry $q/c$ in the first row of $3\phi_1 \left( \frac{q^{-n}, b, q/z}{bq^{1-n}/c}; q, z/c \right)$ was replaced by $q/z$.

*Reported 2016-08-30 by Xinrong Ma.*

**Figures 36.3.9, 36.3.10, 36.3.11, 36.3.12**

Scales were corrected in all figures. The interval $-8.4 \leq \frac{x-y}{\sqrt{2}} \leq 8.4$ was replaced by $-12.0 \leq \frac{x-y}{\sqrt{2}} \leq 12.0$ and $-12.7 \leq \frac{x+y}{\sqrt{2}} \leq 4.2$ replaced by $-18.0 \leq \frac{x+y}{\sqrt{2}} \leq 6.0$. All plots and interactive visualizations were regenerated to improve image quality.

![Figure 36.3.9: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 0)|$.](image-url)
Figure 36.3.10: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 1)|$.

Figure 36.3.11: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 2)|$.

Figure 36.3.12: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 3)|$.

Reported 2016-09-12 by Dan Piponi.
Figures 36.3.18, 36.3.19, 36.3.20, 36.3.21

The scaling error reported on 2016-09-12 by Dan Piponi also applied to contour and density plots for the phase of the hyperbolic umbilic canonical integrals. Scales were corrected in all figures. The interval $-8 < \frac{x-y}{\sqrt{2}} < 8$ was replaced by $-12 < \frac{x-y}{\sqrt{2}} < 12$ and $-18 < \frac{x+y}{\sqrt{2}} < 12$ replaced by $-18 < \frac{x+y}{\sqrt{2}} < 6$. All plots and interactive visualizations were regenerated to improve image quality.

Figure 36.3.18: Phase of hyperbolic umbilic canonical integral $\text{ph } \Psi^{(H)}(x, y, 0)$.

Figure 36.3.19: Phase of hyperbolic umbilic canonical integral $\text{ph } \Psi^{(H)}(x, y, 1)$. 
Figure 36.3.20: Phase of hyperbolic umbilic canonical integral $\text{ph } \Psi^{(H)}(x, y, 2)$.

Figure 36.3.21: Phase of hyperbolic umbilic canonical integral $\text{ph } \Psi^{(H)}(x, y, 3)$.

Other Changes

- A number of changes were made with regard to fractional integrals and derivatives. In §1.15(vi) a reference to Miller and Ross (1993) was added, the fractional integral operator of order $\alpha$ was more precisely identified as the Riemann-Liouville fractional integral operator of order $\alpha$, and a paragraph was added below (1.15.50) to generalize (1.15.47). In §1.15(vii) the sentence defining the fractional derivative was clarified. In §2.6(iii) the identification of the Riemann-Liouville fractional integral operator was made consistent with §1.15(vi).

- Changes to §8.18(ii)–§8.11(v): A sentence was added in §8.18(ii) to refer to Nemes and Olde Daalhuis (2016). Originally §8.11(iii) was applicable for real variables $a$ and $x = \lambda a$. It has been extended to allow for complex variables $a$ and $z = \lambda a$ (and we have replaced $x$ with $z$ in the subsection heading and in Equations (8.11.6) and (8.11.7)). Also, we have added two paragraphs after (8.11.9) to replace the original paragraph that appeared there. Furthermore, the interval of validity of (8.11.6) was increased from $0 < \lambda < 1$ to the sector $0 < \lambda < 1, |\text{ph } a| \leq \frac{\pi}{2} - \delta$, and the interval of validity of (8.11.7) was increased from $\lambda > 1$ to the sector $\lambda > 1, |\text{ph } a| \leq \frac{3\pi}{2} - \delta$. A paragraph with reference to Nemes (2016) has been added in §8.11(v), and the sector of validity for (8.11.12) was increased.
from $|\text{ph } z| \leq \pi - \delta$ to $|\text{ph } z| \leq 2\pi - \delta$. Two new Subsections 13.6(vii), 13.18(vi), both entitled Coulomb Functions, were added to note the relationship of the Kummer and Whittaker functions to various forms of the Coulomb functions. A sentence was added in both §13.10(vi) and §13.23(v) noting that certain generalized orthogonality can be expressed in terms of Kummer functions.

- Four of the terms in (14.15.23) were rewritten for improved clarity.
- In §15.6 it was noted that (15.6.8) can be rewritten as a fractional integral.
- In applying changes in Version 1.0.12 to (16.15.3), an editing error was made; it has been corrected.
- In §34.1, the reference for Clebsch-Gordan coefficients, Condon and Shortley (1935), was replaced by Edmonds (1974) and Rotenberg et al. (1959). The references for 3j, 6j, 9j symbols were made more precise.
- Images in Figures 36.3.1, 36.3.2, 36.3.3, 36.3.4, 36.3.5, 36.3.6, 36.3.7, 36.3.8 and Figures 36.3.13, 36.3.14, 36.3.15, 36.3.16, 36.3.17 were resized for consistency.
- Meta.Numerics (website) was added to the Software Table at http://dlmf.nist.gov/software/.

Version 1.0.13 (September 16, 2016)

Other Changes

- In applying changes in Version 1.0.12 to (13.9.16), an editing error was made; it has been corrected.

---

Version 1.0.12 (September 9, 2016)

Equations (25.11.6), (25.11.19), and (25.11.20)

Originally all six integrands in these equations were incorrect because their numerators contained the function $\tilde{B}_2(x)$. The correct function is $\frac{\tilde{B}_2(x) - B_2}{2}$. The new equations are:

25.11.6

$$\zeta(s,a) = \frac{1}{a^s} \left( \frac{1}{2} + \frac{a}{s-1} \right) - \frac{s(s+1)}{2} \int_0^\infty \frac{\tilde{B}_2(x) - B_2}{(x+a)^{s+2}} \, dx, \quad s \neq 1, \Re s > -1, a > 0$$

Reported 2016-05-08 by Clemens Heuberger.

25.11.19

$$\zeta'(s,a) = -\frac{\ln a}{a^s} \left( \frac{1}{2} + \frac{a}{s-1} \right) - \frac{a^{1-s}}{(s-1)^2} + \frac{s(s+1)}{2} \int_0^\infty \frac{\tilde{B}_2(x) - B_2}{(x+a)^{s+2}} \ln(x+a) \, dx$$

$$- \frac{(2s+1)}{2} \int_0^\infty \frac{\tilde{B}_2(x) - B_2}{(x+a)^{s+2}} \, dx, \quad \Re s > -1, s \neq 1, a > 0$$

Reported 2016-06-27 by Gergő Nemes.

25.11.20

$$(-1)^k \zeta^{(k)}(s,a) = \frac{(\ln a)^k}{a^s} \left( \frac{1}{2} + \frac{a}{s-1} \right) + \frac{k!}{s-1} \sum_{r=0}^{k-1} \frac{(\ln a)^r}{r!(s-1)^{k-r+1}}$$

$$- \frac{s(s+1)}{2} \int_0^\infty \frac{(\tilde{B}_2(x) - B_2)(\ln(x+a))^k}{(x+a)^{s+2}} \, dx + \frac{k(2s+1)}{2} \int_0^\infty \frac{(\tilde{B}_2(x) - B_2)(\ln(x+a))^{k-1}}{(x+a)^{s+2}} \, dx$$

$$- \frac{k(k-1)}{2} \frac{1}{2} \int_0^\infty \frac{(\tilde{B}_2(x) - B_2)(\ln(x+a))^{k-2}}{(x+a)^{s+2}} \, dx, \quad \Re s > -1, s \neq 1, a > 0$$

Reported 2016-06-27 by Gergő Nemes.
Other Changes

- The symbol \( \sim \) is used for two purposes in the DLMF, in some cases for asymptotic equality and in other cases for asymptotic expansion, but links to the appropriate definitions were not provided. In this release changes have been made to provide these links.

- A short paragraph dealing with asymptotic approximations that are expressed in terms of two or more Poincaré asymptotic expansions has been added in §2.1(iii) below (2.1.16).

- Because (2.11.4) is not an asymptotic expansion, the symbol \( \sim \) that was used originally is incorrect and has been replaced with \( \approx \), together with a slight change of wording.

- Originally (13.9.16) was expressed in term of asymptotic symbol \( \sim \). As a consequence of the use of the \( O \) order symbol on the right-hand side, \( \sim \) was replaced by =.

- In (13.9.9) and (13.9.10) there were clarifications made in the conditions on the parameter \( a \) in \( U(a, b, z) \) of those equations.

- Originally (14.15.23) used \( f(x) \) to represent both \( U(-c, x) \) and \( U(-c, x) \). This has been replaced by two equations giving explicit definitions for the two envelope functions. Some slight changes in wording were needed to make this clear to readers.

- The title for §17.9 was changed from Transformations of Higher \( \phi_r \) Functions to Further Transformations of \( r+1 \phi_r \) Functions.

- A number of additions and changes have been made to the metadata in Chapter 25 to reflect new and changed references as well as to how some equations have been derived.

- Bibliographic citations, clarifications, typographical corrections and added or modified sentences appear in §§18.15(i) and 18.16(ii).

Version 1.0.11 (June 8, 2016)

Figure 4.3.1

This figure was rescaled, with symmetry lines added, to make evident the symmetry due to the inverse relationship between the two functions.

Equation (9.7.17)

Originally the constraint, \( \frac{2}{3} \pi \leq |\text{ph} z| < \pi \), was written incorrectly as \( \frac{2}{3} \pi \leq |\text{ph} z| \leq \pi \). Also, the equation was reformatted to display the constraints in the equation instead of in the text.

Reported 2014-11-05 by Gergő Nemes.

Equation (10.32.13)

Originally the constraint, \( |\text{ph} z| < \frac{1}{2} \pi \), was incorrectly written as, \( |\text{ph} z| < \pi \).


Equation (10.40.12)

Originally the third constraint \( \pi \leq |\text{ph} z| < \frac{3}{2} \pi \) was incorrectly written as \( \pi \leq |\text{ph} z| \leq \frac{3}{2} \pi \).

Reported 2014-11-05 by Gergő Nemes.

Equation (23.18.7)

\[ s(d, c) = \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \left[ \frac{dr}{c} \right] - \frac{1}{2} \right), \quad c > 0 \]

Originally the sum \( \sum_{r=1}^{c-1} \) was written with an additional condition on the summation, that \( (r, c) = 1 \). This additional condition was incorrect and has been removed.

Reported 2015-10-05 by Howard Cohl and Tanay Wakhare.
Equations (28.28.21) and (28.28.22)

28.28.21

\[ 4 \pi \int_{0}^{\pi/2} C_{2\ell+1}^{(j)}(2hR) \cos((2\ell + 1)\phi) e_{2m+1}(t, h^2) \, dt \]

\[ = (-1)^{\ell + m + 1} A_{2\ell+1}^{2m+1}(h^2) \mathcal{M}_{2m+1}(z, h) \]

28.28.22

\[ 4 \pi \int_{0}^{\pi/2} C_{2\ell+1}^{(j)}(2hR) \sin((2\ell + 1)\phi) s_{2m+1}(t, h^2) \, dt \]

\[ = (-1)^{\ell + m + 1} B_{2\ell+1}^{2m+1}(h^2) \mathcal{M}_{2m+1}(z, h), \]

Originally the prefactor \( \frac{1}{2} \) and upper limit of integration \( \pi/2 \) in these two equations were given incorrectly as \( \frac{\pi}{2} \) and \( \pi \).

Reported 2015-05-20 by Ruslan Kabasayev

Other Changes

- In §1.2(i), a sentence was added after (1.2.1) to refer to (1.2.6) as the definition of the binomial coefficient \( \binom{z}{n} \) when \( z \) is complex. As a notational clarification, wherever \( n \) appeared originally in (1.2.6)–(1.2.9), it was replaced by \( z \).

- It was reported by Nico Temme on 2015-02-28 that the asymptotic formula for \( \ln \Gamma(z + h) \) given in (5.11.8) is valid for \( h \in \mathbb{C} \); originally it was unnecessarily restricted to \([0, 1]\).

- In §13.8(iii), a new paragraph with several new equations and a new reference has been added at the end to provide asymptotic expansions for Kummer functions \( U(a, b, z) \) and \( M(a, b, z) \) as \( a \to \infty \) in \( |\text{ph}\, a| \leq \pi - \delta \) and \( b \) and \( z \) fixed.

- Because of the use of the \( O \) order symbol on the right-hand side, the asymptotic expansion (18.15.22) for the generalized Laguerre polynomial \( L_n^{(\alpha)}(\nu x) \) was rewritten as an equality.

- The entire Section 27.20 was replaced.

- Bibliographic citations have been added or modified in §§2.4(v), 2.4(vi), 2.9(iii), 5.11(i), 5.11(ii), 5.17, 9.9(i), 10.22(v), 10.37, 11.6(iii), 11.9(iii), 12.9(ii), 13.8(ii), 13.11, 14.15(i), 14.15(ii), 15.12(ii), 15.14, 16.11(ii), 16.13, 18.15(vi), 20.7(viii), 24.11, 24.16(i), 26.8(vii), 33.12(i), and 33.12(ii).

- Clarifications, typographic corrections, added or modified sentences appear in §§1.2(i), 1.10(i), 4.6(ii), 5.11(i), (11.11.1), (11.11.9), (21.5.7), and (27.14.7).

Version 1.0.10 (August 7, 2015)

Section 4.43

The first paragraph has been rewritten to correct reported errors. The new version is reproduced here.

Let \( p (\neq 0) \) and \( q \) be real constants and

\[ A = (-\frac{4}{3}p)^{1/2}, \quad B = (\frac{4}{3}p)^{1/2}. \]

The roots of

\[ z^3 + pz + q = 0 \]

are:

- (a) \( A \sin a, \ A \sin(a + \frac{\pi}{3}), \) and \( A \sin(a + \frac{2\pi}{3}) \), with \( \sin(3a) = 4q/A^3 \), when \( 4p^3 + 27q^2 \leq 0 \).

- (b) \( A \cosh a, \ A \cosh(a + \frac{\pi}{3}), \) and \( A \cosh(a + \frac{2\pi}{3}) \), with \( \cosh(3a) = -4q/A^3 \), when \( p < 0, q < 0, \) and \( 4p^3 + 27q^2 > 0 \).

- (c) \( B \sinh a, \ B \sinh(a + \frac{\pi}{3}), \) and \( B \sinh(a + \frac{2\pi}{3}) \), with \( \sinh(3a) = -4q/B^3 \), when \( p > 0 \).

Note that in Case (a) all the roots are real, whereas in Cases (b) and (c) there is one real root and a conjugate pair of complex roots. See also §1.11(iii).

Reported 2014-10-31 by Masataka Urago.

Equation (9.10.18)

\[ \text{Ai}(z) = \frac{3z^{5/4}e^{-(2/3)z^{3/2}}}{4\pi} \times \int_{0}^{\infty} \frac{t^{-3/4}e^{-(2/3)t^{3/2}}\text{Ai}(t)}{z^{3/2} + t^{3/2}} \, dt \]

The original equation taken from Schulten et al. (1979) was incorrect.

Reported 2015-03-20 by Walter Gautschi.

Equation (9.10.19)

\[ \text{Bi}(x) = \frac{3x^{5/4}e^{(2/3)x^{3/2}}}{2\pi} \times \int_{0}^{\infty} \frac{t^{-3/4}e^{-(2/3)t^{3/2}}\text{Ai}(t)}{x^{3/2} - t^{3/2}} \, dt \]

The original equation taken from Schulten et al. (1979) was incorrect.

Reported 2015-03-20 by Walter Gautschi.
Equation (10.17.14)

10.17.14 \[ |R_\nu^\ell(v, z)| \leq 2|\alpha(v)|\sqrt{\nu, \pm\infty}(t^{-\nu}) \times \exp \left( (\nu^2 - \frac{1}{2})\sqrt{\nu, \pm\infty}(t^{-1}) \right) \]

Originally the factor \( \sqrt{\nu, \pm\infty}(t^{-1}) \) in the argument to the exponential was written incorrectly as \( \sqrt{\nu, \pm\infty}(t^{-\nu}) \).

Reported 2014-09-27 by Gergő Nemes.

Equation (10.19.11)

10.19.11 \[ Q_3(a) = \frac{549}{28000}a^8 - \frac{1}{10767}a^5 + \frac{79}{12175}a^2 \]

Originally the first term on the right-hand side of this equation was written incorrectly as \(-\frac{549}{28000}a^8\).

Reported 2015-03-16 by Svante Janson.

Equation (13.2.7)

13.2.7 \[ U(-m, b, z) = (-1)^m(b)_mM(-m, b, z) \]

\[ U(-m, b, z) = (-1)^m \sum_{s=0}^{m} \binom{m}{s} (b + s)_{m-s}(-z)^s \]

The equality \( U(-m, b, z) = (-1)^m(b)_mM(-m, b, z) \) has been added to the original equation to express an explicit connection between the two standard solutions of Kummer’s equation. Note also that the notation \( a = -n \) has been changed to \( a = -m \).

Reported 2015-02-10 by Adri Olde Daalhuis.

Equation (13.2.8)

13.2.8 \[ U(a, a + n + 1, z) = \frac{(-1)^n(1 - a - n)_n}{z^{a+n}} \times M(-n, 1 - a - n, z) \]

\[ U(a, a + n + 1, z) = z^{-a} \sum_{s=0}^{n} \binom{n}{s} (a)_s z^{-s} \]

The equality \( U(a, a + n + 1, z) = \frac{(-1)^n(1 - a - n)_n}{z^{a+n}} \times M(-n, 1 - a - n, z) \) has been added to the original equation to express an explicit connection between the two standard solutions of Kummer’s equation.

Reported 2015-02-10 by Adri Olde Daalhuis.

Equation (13.2.10)

13.2.10 \[ U(-m, n + 1, z) \]

\[ U(-m, n + 1, z) = (-1)^m(n + 1)_mM(-m, n + 1, z) \]

\[ U(-m, n + 1, z) = (-1)^m \sum_{s=0}^{m} \binom{m}{s} (n + s + 1)_{m-s}(-z)^s \]

The equality \( U(-m, n + 1, z) = (-1)^m(n + 1)_m \times M(-m, n + 1, z) \) has been added to the original equation to express an explicit connection between the two standard solutions of Kummer’s equation. Note also that the notation \( a = -m, m = 0, 1, 2, \ldots \) has been introduced.

Reported 2015-02-10 by Adri Olde Daalhuis.

Equation (18.33.3)

18.33.3 \[ \phi_n(z) = z^n\phi_n(z^{-1}) = \kappa_n + \sum_{\ell=1}^{n} R_{n,n-\ell} z^\ell \]

Originally this equation was written incorrectly as \( \phi_n(z) = \kappa_n z^n + \sum_{\ell=1}^{n} R_{n,n-\ell} z^{n-\ell} \). Also, the equality \( \phi_n(z) = z^n\phi_n(z^{-1}) \) has been added.

Reported 2014-10-03 by Roderick Wong.

Equation (34.7.4)

34.7.4 \[ \left( \begin{array}{ccc} j_{13} & j_{23} & j_{33} \\ m_{13} & m_{23} & m_{33} \end{array} \right) = \sum_{m_{11},m_{12},r_{1}=1,2,3} \left( \begin{array}{ccc} j_{11} & j_{12} & j_{13} \\ m_{11} & m_{12} & m_{13} \end{array} \right) \times \left( \begin{array}{ccc} j_{21} & j_{22} & j_{23} \\ m_{21} & m_{22} & m_{23} \end{array} \right) \times \left( \begin{array}{ccc} j_{31} & j_{32} & j_{33} \\ m_{31} & m_{32} & m_{33} \end{array} \right) \]

Originally the third \( 3j \) symbol in the summation was written incorrectly as \( \left( \begin{array}{ccc} j_{31} & j_{32} & j_{33} \\ m_{31} & m_{32} & m_{33} \end{array} \right) \).

Reported 2015-01-19 by Yan-Rui Liu.

Other Changes

- To increase the regions of validity (5.9.10), (5.9.11), (5.10.1), (5.11.1), and (5.11.8), the logarithms of the gamma function that appears on their left-hand sides have all been changed to \( \ln \Gamma(z) \), where \( \ln \) is the general logarithm. Originally \( \ln \Gamma(z) \) was used, where \( \ln \) is the principal branch of the logarithm. These changes were recommended by Philippe Spindel on 2015-02-06.

- The notation used for the \( q \)-Appell functions in Section 17.1 and Equations (17.4.5), (17.4.6), (17.4.7), (17.4.8), (17.11.1), (17.11.2), and (17.11.3) was updated to explicitly include the argument \( q \), as used in Gasper and Rahman (2004).
• A note was added after (22.20.5) to deal with cases when computation of \( \text{dn}(x, k) \) becomes numerically unstable near \( x = K \).

• The spelling of the name Delannoy was corrected in several places in §26.6. Previously it was misspelled as Dellanoy.

• For consistency of notation across all chapters, the notation for logarithm has been changed to \( \ln \) from \( \log \) throughout Chapters, the notation for logarithm has been changed to \( \ln \) from \( \log \) throughout Chapters 5–10. The spelling of the name Delannoy was corrected in several places in §26.6. Previously it was misspelled as Dellanoy.

• Bibliographic citations have been added or modified in §§2.4(vi), 3.8(v), 5.6(i), 5.10, 5.11(i), 5.11(ii), 5.18(ii), 7.21, 8.10, 10.21(ix), 10.45, 10.74(vi), 11.7(v), 13.7(ii), 14.17(iii), 14.20(ix), 14.28(ii), 14.32, 15.8(v), 15.13, 15.19(ii), 16.6, 16.13, 17.6(ii), 17.7(iii), 18.1(iii), 18.3, 18.15(iv) and 18.24.

Version 1.0.9 (August 29, 2014)

Equation (9.6.26)

\[
\text{Bi}'(z) = \frac{3^{1/6}}{\Gamma(2/3)} e^{-\zeta} F_1\left(-\frac{1}{2}; -\frac{1}{2}; 2\zeta\right) + \frac{3^{7/6}}{2^{7/3} \Gamma(3/4)} z^{4/3} e^{-\zeta} F_1\left(\frac{5}{6}; \frac{5}{3}; 2\zeta\right)
\]

Equation (34.3.7)

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_1 & -j_1 - m_3 & m_3
\end{pmatrix}
= (-1)^{j_1 - j_2 - m_3} (2j_1)!(j_1 + j_2 + j_3)!(j_1 + j_2 + m_3)!(j_3 - m_3)! \left( \frac{U_{j_1 + j_2 + j_3 + 1} U_{j_1 - j_2 - j_3}}{j_1 + j_2 + j_3} \right)^{1/2}
\]

In the original equation the prefactor of the above 3j symbol read \((-1)^{-j_2 + j_3 + m_3}\). It is now replaced by its correct value \((-1)^{j_1 - j_2 - m_3}\).

Reported 2014-05-02 by Svante Janson.

Paragraph Case III: \( V(x) = -\frac{1}{2}x^2 + \frac{1}{4}\beta x^4 \) in §22.19(ii)

Two corrections have been made in this paragraph. First, the correct range of the initial displacement \( a \) is \( \sqrt{1/\beta} \leq |a| < \sqrt{2/\beta} \). Previously it was \( \sqrt{1/\beta} \leq |a| \leq \sqrt{2/\beta} \). Second, the correct period of the oscillations is \( 2K(k)/\sqrt{\eta} \). Previously it was given incorrectly as \( 4K(k)/\sqrt{\eta} \).

Reported 2014-05-02 by Svante Janson.

Other Changes

• Pochhammer symbols have been introduced in Equations (7.12.1), (7.12.2), (7.12.3), (7.12.4), (7.12.5), (25.5.7), (25.8.3), (25.11.10), (25.11.28), and (25.11.43) to make the notation more concise.

• The Wronskian (14.2.7) was generalized to include both associated Legendre and Ferrers functions.

• A cross-reference has been added in

\[\text{§15.9(iv).}\]

• Equations (22.19.6), (22.19.7), (22.19.8), and (22.19.9) have been rewritten with the modulus (second argument) of the Jacobian elliptic function defined explicitly in the preceding line of text.

• Bibliographic citations have been added in §§4.13, 4.48(iv), 6.21(ii), 8.28(ii), 9.16, 10.77(viii), 12.21(ii), 14.28(ii), 14.34(ii), 77 and 16.13.

• An addition was made to the Software Table...
at http://dlmf.nist.gov/software/ to reflect the addition of a multiple precision (MP) package written in C++ which uses a variety of different MP interfaces.

Version 1.0.8 (April 25, 2014)

Equation (22.19.2)

\[ 22.19.2 \quad \sin \left( \frac{1}{2} \theta(t) \right) = \sin \left( \frac{1}{2} \alpha \right) \text{sn}(t + K, \sin \left( \frac{1}{2} \alpha \right)) \]

Originally the first argument to the function \( \text{sn} \) was given incorrectly as \( t \). The correct argument is \( t + K \).

Reported 2014-03-05 by Svante Janson.

Equation (22.19.3)

\[ 22.19.3 \quad \theta(t) = 2 \text{am} \left( t \sqrt{\frac{E}{2}}, \sqrt{\frac{2}{E}} \right) \]

Originally the first argument to the function \( \text{am} \) was given incorrectly as \( t \). The correct argument is \( t \sqrt{E/2} \).

Reported 2014-03-05 by Svante Janson.

Other Changes

- Minor additions have been made in §§9.6(iii), 22.19(i).
- Equation (10.13.4) has been generalized to cover an additional case.
- We avoid the troublesome symbols, often missing in installed fonts, previously used for exponential \( e \), imaginary \( i \) and differential \( d \).

Version 1.0.7 (March 21, 2014)

Table 3.5.19

The correct headings for the second and third columns of this table are \( J_0(t) \) and \( g(t) \), respectively. Previously these columns were mislabeled as \( g(t) \) and \( J_0(t) \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( J_0(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000 00000</td>
<td>1.00000 00000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.93846 98072</td>
<td>0.93846 98072</td>
</tr>
<tr>
<td>1.0</td>
<td>0.76519 76866</td>
<td>0.76519 76865</td>
</tr>
<tr>
<td>2.0</td>
<td>0.22389 07791</td>
<td>0.22389 10326</td>
</tr>
<tr>
<td>5.0</td>
<td>-0.17759 6713</td>
<td>-0.17902 54097</td>
</tr>
<tr>
<td>10.0</td>
<td>-0.24593 57645</td>
<td>-0.07540 53543</td>
</tr>
</tbody>
</table>

Reported 2014-01-31 by Masataka Urago.

Table 3.5.21

The correct corner coordinates for the 9-point square, given on the last line of this table, are \( (\pm \sqrt{\frac{3}{5}} h, \pm \sqrt{\frac{5}{3}} h) \). Originally they were given incorrectly as \( (\pm \sqrt{\frac{3}{5}} h, 0), (\pm \sqrt{\frac{5}{3}} h, 0) \).

<table>
<thead>
<tr>
<th>Diagram</th>
<th>( (x_j, y_j) )</th>
<th>( w_j )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column 1</td>
<td>Column 2</td>
<td>Column 3</td>
<td>Column 4</td>
</tr>
<tr>
<td>(0,0)</td>
<td>(0,±h)</td>
<td>(0,±h)</td>
<td>0.5</td>
</tr>
<tr>
<td>(±1/2h,±1/2h)</td>
<td>1/4</td>
<td>Ω(( h^4 ))</td>
<td></td>
</tr>
<tr>
<td>(0,0)</td>
<td>(±h,0), (0,±h)</td>
<td>(±1/2h,±1/2h)</td>
<td>1/6</td>
</tr>
<tr>
<td>(0,0)</td>
<td>(±1/6\sqrt{6}h,0)</td>
<td>(±1/6\sqrt{6}h,±1/2\sqrt{2}h)</td>
<td>1/6</td>
</tr>
<tr>
<td>(0,0)</td>
<td>(±1/3\sqrt{3}h,±1/3\sqrt{3}h)</td>
<td>1/4</td>
<td>Ω(( h^4 ))</td>
</tr>
<tr>
<td>(0,0)</td>
<td>(±\sqrt{3/5}h,0), (0,±\sqrt{3/5}h)</td>
<td>(±\sqrt{3/5}h,±\sqrt{3/5}h)</td>
<td>16/21</td>
</tr>
</tbody>
</table>

Equation (4.21.1)

\[ \sin u \pm \cos u = \sqrt{2} \sin \left( u \pm \frac{1}{4} \pi \right) = \pm \sqrt{2} \cos \left( u \pm \frac{1}{4} \pi \right) \]

Originally the symbol ± was missing after the second equal sign.

Reported 2012-09-27 by Dennis Heim.

Equations (4.23.34) and (4.23.35)

\[ \arcsin z = \arcsin \beta + i \text{sign}(y) \ln \left( \alpha + (\alpha^2 - 1)^{1/2} \right) \]

and

\[ \arccos z = \arccos \beta - i \text{sign}(y) \ln \left( \alpha + (\alpha^2 - 1)^{1/2} \right) \]

Originally the factor sign(y) was missing from the second term on the right sides of these equations. Additionally, the condition for the validity of these equations has been weakened.

Reported 2013-07-01 by Volker Thürey.

Equation (5.17.5)

\[ \ln G(z + 1) \sim \frac{1}{4} z^2 + z \ln \Gamma(z + 1) - \left( \frac{1}{2} z(z + 1) + \frac{1}{12} \right) \ln z - \ln A + \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)} z^{2k} \]

Originally the term \( z \ln \Gamma(z + 1) \) was incorrectly stated as \( z \Gamma(z + 1) \).

Reported 2013-08-01 by Gergő Nemes and subsequently by Nick Jones on December 11, 2013.

Table 22.4.3

A correction was made in the online portion of this table.

Reported 2014-02-28 by Svante Janson.

Table 22.5.2

The entry for \( \text{sn} z \) at \( z = \frac{3}{2}(K+iK') \) has been corrected. The correct entry is \((1+i)((1+k')^{1/2} - i(1-k')^{1/2})/(2k')\). Originally the terms \( (1+k')^{1/2} \) and \((1-k')^{1/2} \) were given incorrectly as \((1+k)^{1/2} \) and \((1-k)^{1/2} \).

Similarly, the entry for \( \text{dn} z \) at \( z = \frac{3}{2}(K+iK') \) has been corrected. The correct entry is \((-1 + i)k^{1/2}((1+k)^{1/2} + i(1-k)^{1/2})/2 \). Originally the terms \((1+k)^{1/2} \) and \((1-k)^{1/2} \) were given incorrectly as \((1+k')^{1/2} \) and \((1-k')^{1/2} \).

Reported 2014-02-28 by Svante Janson.

Equation (22.6.7)

\[ \text{dn}(2z,k) = \frac{\text{dn}^2(z,k) - k^2 \text{sn}^2(z,k) \text{cn}^2(z,k)}{1 - k^2 \text{sn}^4(z,k)} \]

\[ = \frac{\text{dn}^4(z,k) + k^2 k'^2 \text{sn}^4(z,k)}{1 - k^2 \text{sn}^4(z,k)} \]

Originally the term \( k^2 \text{sn}^2(z,k) \text{cn}^2(z,k) \) was given incorrectly as \( k^2 \text{sn}^2(z,k) \text{dn}^2(z,k) \).

Reported 2014-02-28 by Svante Janson.

Table 26.8.1

Originally the Stirling number \( s(10,6) \) was given incorrectly as 6327. The correct number is 63273.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>-3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-6</td>
<td>11</td>
<td>-6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>24</td>
<td>-50</td>
<td>35</td>
<td>-10</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>-120</td>
<td>274</td>
<td>-225</td>
<td>85</td>
<td>-15</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>720</td>
<td>-1764</td>
<td>1624</td>
<td>-735</td>
<td>175</td>
<td>-21</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>-5040</td>
<td>13068</td>
<td>-13132</td>
<td>6769</td>
<td>-1960</td>
<td>322</td>
<td>-28</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>40320</td>
<td>-109584</td>
<td>118124</td>
<td>-67284</td>
<td>22449</td>
<td>-4536</td>
<td>546</td>
<td>-36</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>-362880</td>
<td>1026576</td>
<td>-1172700</td>
<td>723680</td>
<td>-269325</td>
<td>63273</td>
<td>-9450</td>
<td>870</td>
<td>-45</td>
<td>1</td>
</tr>
</tbody>
</table>
Equation (31.8.5)

31.8.5 \( \Psi_{1,-1} = \left( z^2 + (\lambda + 3a + 3)z + a \right) / z^3 \)

Originally the first term on the right side of the equation for \( \Psi_{1,-1} \) was \( z^3 \). The correct factor is \( z^2 \).

Reported 2013-07-25 by Christopher Künstler.

Equation (31.12.3)

31.12.3 \( \frac{d^2 w}{dz^2} - \left( \frac{\gamma}{z} + \delta + z \right) \frac{dw}{dz} + \frac{\alpha z - q}{z} w = 0 \)

Originally the sign in front of the second term in this equation was +. The correct sign is −.

Reported 2013-10-31 by Henryk Witek.

Equation (34.4.2)

34.4.2 \[ \{ j_1 \ j_2 \ j_3 \} \Delta (j_1 j_2 j_3) \Delta (j_1 l_2 l_3) \Delta (l_1 j_2 l_3) \Delta (l_1 l_2 j_3) \]

\[ \times \sum_s \frac{(-1)^s (s + 1)!}{(s - j_1 - j_2 - j_3)! (s - j_1 - l_2 - l_3)! (s - l_1 - j_2 - l_3)! (s - l_1 - l_2 - j_3)!} \]

\[ \times \frac{(j_1 + j_2 + l_1 + l_2 - s)! (j_2 + j_3 + l_2 + l_3 - s)!(j_3 + j_1 + l_3 + l_1 - s)!}{(j_1 + j_2 + j_3)! (j_1 + l_2 + l_3)! (j_2 + j_3 + l_1)! (j_3 + j_1 + l_3)!} \]

Originally the factor \( \Delta (j_1 j_2 j_3) \Delta (j_1 l_2 l_3) \Delta (l_1 j_2 l_3) \Delta (l_1 l_2 j_3) \) was missing in this equation.

Reported 2012-12-31 by Yu Lin.

Other Changes

- Equations (4.45.8) and (4.45.9) have been replaced with equations that are better for numerically computing \( \arctan x \).
- A new Subsection 13.29(v) Continued Fractions, has been added to cover computation of confluent hypergeometric functions by continued fractions.
- A new Subsection 14.5(vi) Addendum to §14.5(ii) \( \mu = 0, \nu = 2 \), containing the values of Legendre and Ferrers functions for degree \( \nu = 2 \) has been added.
- Subsection 14.18(iii) has been altered to identify Equations (14.18.6) and (14.18.7) as Christoffel’s Formulas.
- A new Subsection 15.19(v) Continued Fractions, has been added to cover computation of the Gauss hypergeometric functions by continued fractions.
- Special cases of normalization of Jacobi polynomials for which the general formula is undefined have been stated explicitly in Table 18.3.1.
- Bibliographic citations have been added in §§4.13, 5.6(i), 5.11(iii), 7.25(iii), 8.13(i), 10.37, 12.18, 14.11, 15.12(ii), 16.6, 18.16(ii), 18.16(iv), 18.24, 18.27(iv), 18.27(v),18.28(i), 24.13(i), 28.36(iii).
- Cross-references have been added in §§1.2(i), 10.19(iii), 10.23(ii), 17.2(iii), 18.15(iii), 19.2(iv), 19.16(i).
- Several small revisions have been made. For details see §§3.11(ii), 10.12, 10.19(ii), 18.9(i), 18.16(iv), 19.7(ii), 22.2, 32.11(v), 32.13(ii).
- Entries for the Sage computational system have been updated in the online Software Cross Index.
- The default document format for DLMF is now HTML5 which includes MathML providing better accessibility and display of mathematics.
- All interactive 3D graphics on the DLMF website have been recast using WebGL and X3DOM, improving portability and performance; WebGL it is now the default format.

Version 1.0.6 (May 6, 2013)

Several minor improvements were made affecting display and layout; primarily tracking changes to the un-
derlying LaTeX system.

Version 1.0.5 (October 1, 2012)

Subsection 1.2(i)

The condition for (1.2.2), (1.2.4), and (1.2.5) was corrected. These equations are true only if \( n \) is a positive integer. Previously \( n \) was allowed to be zero.

Reported 2011-08-10 by Michael Somos.

Subsection 8.17(i)

The condition for the validity of (8.17.5) is that \( m \) and \( n \) are positive integers and \( 0 \leq x < 1 \). Previously, no conditions were stated.

Reported 2011-03-23 by Stephen Bourn.

Equation (10.20.14)

\[
\begin{align*}
B_3(0) &= -\frac{959\,717\,111\,846\,032\,547\,666\,371\,250\,000\,000}{2^{13/2}} \\
\end{align*}
\]

Originally this coefficient was given incorrectly as \( B_3(0) = -\frac{430\,990\,563\,936\,859\,253\,568\,167\,343\,994\,250\,000\,000}{2^{13/2}} \). The other coefficients in this equation have not been changed.

Reported 2012-05-11 by Antony Lee.

Equation (13.16.4)

\[
\Re(\kappa - \mu) - \frac{1}{2} < 0.
\]

Originally it was given incorrectly as \( \Re(\kappa - \mu) - \frac{1}{2} > 0 \).

Reported 2012-07-18 by Hans Volkmer and Howard Cohl.

Subsection 14.2(ii)

Originally it was stated, incorrectly, that \( Q_{\nu}^\mu(x) \) is real when \( \nu, \mu \in \mathbb{R} \) and \( x \in (1, \infty) \). This statement is true only for \( P_{\nu}^\mu(x) \) and \( Q_{\nu}^\mu(x) \).

Reported 2012-07-18 by Hans Volkmer and Howard Cohl.

Equation (21.3.4)

\[
\begin{align*}
\theta\left[ \begin{array}{c} \alpha + m_1 \\ \beta + m_2 \end{array} \right](z|\Omega) &= e^{2\pi i \alpha \cdot m_2} \theta\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right](z|\Omega)
\end{align*}
\]

Originally the vector \( m_2 \) on the right-hand side was given incorrectly as \( m_1 \).

Reported 2012-08-27 by Klaas Vantournhout.

Subsection 21.10(i)

The entire original content of this subsection has been replaced by a reference.

Figures 22.3.22 and 22.3.23

The captions for these figures have been corrected to read, in part, “as a function of \( k^2 = i\kappa^2 \)” instead of \( k^2 = i\kappa \). Also, the resolution of the graph in Figure 22.3.22 was improved near \( \kappa = 3 \).

Reported 2011-10-30 by Paul Abbott.

Equation (23.2.4)

\[
\varphi(z) = \frac{1}{z^2} + \sum_{w \in L \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)
\]

Originally the denominator \( (z-w)^2 \) was given incorrectly as \( (z-w)^2 \).

Reported 2012-02-16 by James D. Walker.

Equation (24.4.26)

This equation is true only for \( n > 0 \). Previously, \( n = 0 \) was also allowed.

Reported 2012-05-14 by Vladimir Yurovsky.

Equation (26.12.26)

\[
\text{pp}(n) \sim \frac{\zeta(3)^{7/36}}{2^{11/36}(3\pi)^{1/2} n^{25/36}} \times \exp\left( 3 \left( \frac{\zeta(3)}{2} \right)^{1/3} \left( \frac{1}{2} n \right)^{2/3} + \zeta'(-1) \right)
\]

Originally this equation was given incorrectly as

\[
\text{pp}(n) \sim \frac{\zeta(3)^{1/3}}{2^{11/36} n^{25/36}} \times \exp\left( 3 \left( \frac{\zeta(3)n^2}{4} \right)^{1/3} + \zeta'(-1) \right)
\]

Reported 2011-09-05 by Suresh Govindarajan.

Other Changes

- On August 24, 2012 Dr. Adri B. Olde Daalhuis was added as Mathematics Editor. This addition has been recorded at the end of the Preface (p. ix et seq.)
- Bibliographic citations were added in §§5.5(iii), 5.6(i), 5.10, 5.21, 7.13(ii), 10.19(iii), 10.21(i), 10.21(iv), 10.21(xiii), 10.21(xiv), 10.42, 10.46, 10.74(vii), 13.8(ii),
Errata

13.9(i), 13.9(ii), 11.13, 13.29(iv), 14.11, 15.13, 15.19(i), 17.18, 18.16(ii), 18.16(iv), 18.26(v), 19.12, 19.36(iv), 20.7(i), 20.7(ii), 20.7(iii), 20.7(vii), 25.11(iv), 25.18(i), 26.12(iv), 28.24, 28.34(ii), 29.20(i), 31.17(ii), 32.17, and as a general reference in Chapter 3.

• A cross-reference was added in §21.2(i).
• Several new equations have been added. See (8.17.24), (20.7.34), §20.11(v), (26.12.27), (36.2.28), and (36.2.29).
• The upper and lower bounds given in Equations (18.16.12) and (18.16.13) have been replaced with stronger bounds.
• Textual clarifications were made in §§1.5(ii), 7.13(ii), 15.6, 19.12, 20.7(iv), 21.2(i), 30.13(i), 30.14(i), and 31.17(ii).
• Other minor changes were made in the bibliography and index.

Version 1.0.4 (March 23, 2012)

Several minor improvements were made affecting display of math and graphics on the website; the software index and help files were updated.

Version 1.0.3 (Aug 29, 2011)

Equation (13.18.7)

13.18.7 \[ W_{-\frac{1}{4}, \pm \frac{1}{4}} (z^2) = e^{\frac{1}{2} z^2} \sqrt{\pi} \text{erfc}(z) \]

Originally the left-hand side was given correctly as \( W_{-\frac{1}{4}, -\frac{1}{4}} (z^2) \); the equation is true also for \( W_{-\frac{1}{4}, +\frac{1}{4}} (z^2) \).

Other Changes

Bibliographic citations were added in §§3.5(iv), 4.44, 8.22(ii), 22.4(i), and minor clarifications were made in §§19.12, 20.7(vii), 22.9(i). In addition, several minor improvements were made affecting only ancilliary documents and links in the online version.

Version 1.0.2 (July 1, 2011)

Several minor improvements were made affecting display on the website; the help files were revised.

Version 1.0.1 (June 27, 2011)

Subsections 1.15(vi) and 1.15(vii)

The formulas in these subsections are valid only for \( x \geq 0 \). No conditions on \( x \) were given originally.

Reported 2010-10-18 by Andreas Kurt Richter.

Figure 10.48.5

Originally the ordinate labels 2 and 4 in this figure were placed too high.

Reported 2010-11-08 by Wolfgang Ehrhardt.

Equation (14.19.2)

\[ P_{\nu - \frac{1}{2}}^{\mu} (\cosh \xi) = \frac{\Gamma(\frac{1}{2} - \mu)}{\pi^{1/2} (1 - e^{-2\xi})^\mu e^{(\nu+(1/2))\xi}} \times F\left(\frac{1}{2} - \mu, \frac{1}{2} + \nu - \mu; 1 - 2\mu; 1 - e^{-2\xi}\right), \]

\[ \mu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \]

Originally the argument to \( F \) in this equation was incorrect \((e^{-2\xi}, \) rather than \(1 - e^{-2\xi})\), and the condition on \( \mu \) was too weak \((\mu \neq \frac{1}{2}, \) rather than \(\mu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\)). Also, the factor multiplying \( F \) was rewritten to clarify the poles; originally it was \( \Gamma(1-2\mu)/(1-e^{-2\xi})^2 e^{(\nu+1/2)\xi} \).

Reported 2010-11-02 by Alvaro Valenzuela.

Equation (17.13.3)

\[ \int_0^\infty t^{\alpha-1} (-tq^{\alpha+\beta}; q)_\infty \, dt \]

\[ = \frac{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(\beta)}{\Gamma_q(1-\alpha)\Gamma_q(\alpha + \beta)} \]

Originally the differential was identified incorrectly as \( dq/t \); the correct differential is \( dt \).

Reported 2011-04-08.
Table 18.9.1

The coefficient $A_n$ for $C_n^{(\lambda)}(x)$ in the first row of this table originally omitted the parentheses and was given as $\frac{2n+\lambda}{n+1} B_n$, instead of $\frac{2(n+\lambda)}{n+1}$.  

<table>
<thead>
<tr>
<th>$p_n(x)$</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_n^{(\lambda)}(x)$</td>
<td>$\frac{2(n+\lambda)}{n+1}$</td>
<td>$0$</td>
<td>$\frac{n+2\lambda-1}{n+1}$</td>
</tr>
<tr>
<td>$T_n(x)$</td>
<td>$2 - \delta_{n,0}$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$U_n(x)$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$T^*_n(x)$</td>
<td>$4 - 2\delta_{n,0}$</td>
<td>$-2 + \delta_{n,0}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$U^*_n(x)$</td>
<td>$4$</td>
<td>$-2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$P_n(x)$</td>
<td>$\frac{2n+1}{n+1}$</td>
<td>$0$</td>
<td>$\frac{n}{n+1}$</td>
</tr>
<tr>
<td>$P^*_n(x)$</td>
<td>$\frac{4n+2}{n+1}$</td>
<td>$-\frac{2n+1}{n+1}$</td>
<td>$\frac{n}{n+1}$</td>
</tr>
<tr>
<td>$L_n^{(\alpha)}(x)$</td>
<td>$-\frac{1}{n+1}$</td>
<td>$\frac{2n+\alpha+1}{n+1}$</td>
<td>$\frac{n+\alpha}{n+1}$</td>
</tr>
<tr>
<td>$H_n(x)$</td>
<td>$2$</td>
<td>$0$</td>
<td>$2n$</td>
</tr>
<tr>
<td>$He_n(x)$</td>
<td>$1$</td>
<td>$0$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Reported 2010-09-16 by Kendall Atkinson.

Subsection 19.16(iii)

Originally it was implied that $R_C(x,y)$ is an elliptic integral. It was clarified that $R_{-a}(b;z)$ is an elliptic integral iff the stated conditions hold; originally these conditions were stated as sufficient but not necessary. In particular, $R_C(x,y)$ does not satisfy these conditions.

Reported 2010-11-23.

Table 22.5.4

Originally the limiting form for $sc(z,k)$ in the last line of this table was incorrect ($\cosh z$, instead of $\sinh z$).

| $sn(z,k)$ | $\to$ | $\tanh z$ |
| $cn(z,k)$ | $\to$ | $\sech z$ |
| $dn(z,k)$ | $\to$ | $\sech z$ |
| $cd(z,k)$ | $\to$ | $1$ |
| $sd(z,k)$ | $\to$ | $\sinh z$ |
| $nd(z,k)$ | $\to$ | $\cosh z$ |
| $dc(z,k)$ | $\to$ | $1$ |
| $nc(z,k)$ | $\to$ | $\cosh z$ |
| $ns(z,k)$ | $\to$ | $\coth z$ |
| $ds(z,k)$ | $\to$ | $\csch z$ |
| $sc(z,k)$ | $\to$ | $\sinh z$ |
| $cs(z,k)$ | $\to$ | $\csch z$ |

Reported 2010-11-23.

Equation (22.16.14)

Originally this equation appeared with the upper limit of integration as $x$, rather than $sn(x,k)$.

Reported 2010-07-08 by Charles Karney.

Equation (26.7.6)

Originally this equation appeared with $B(n)$ in the summation, instead of $B(k)$.

Reported 2010-11-07 by Layne Watson.

Equation (36.10.14)

Originally this equation appeared with $\frac{\partial^2 \Psi^{(E)}}{\partial x^2} - \frac{\partial^2 \Psi^{(E)}}{\partial y^2}$ in the second term, rather than $\frac{\partial^2 \Psi^{(E)}}{\partial x^2}$.

Reported 2010-04-02.
Other Changes

- The definition of the notation $F(z_0 e^{2k\pi i})$ was added in Common Notations and Definitions on p. xiv.

- Clarifications were made in §§5.18, 7.1, 9.2(iii), 10.15, 10.38, 14.13, 15.8(i), 15.10(i), 16.11(ii), 19.2(iv), 19.16(ii), 19.16(iii), 22.16(ii), 27.16.

- Bibliographic citations were added in §§1.13(v), 10.14, 10.21(ii), 18.15(v), 18.32, 30.16(iii), 32.13(ii), and as general references in Chapters 19, 20, 22, and 23.

- The general references for each chapter were inserted under the $i$-symbol on the chapter title pages. Originally these appeared only in the References sections of the individual chapters in the Handbook.

- The definition of $R_C(x, y)$ was revised in Notations beginning on p. 909.

- Additions and revisions were made in the Cross Index for Computing Special Functions at http://dlmf.nist.gov/software/.

To see the effect of these changes, see http://dlmf.nist.gov/.

Version 1.0.0 (May 7, 2010)

The Handbook of Mathematical Functions was published, and the Digital Library of Mathematical Functions was released.

Bibliography


