Errata

The following corrections and other changes have been made at http://dlmf.nist.gov/, and are pending for the Handbook of Mathematical Functions. The Editors thank the users who have contributed to the accuracy of the DLMF Project by submitting reports of possible errors. For confirmed errors, the Editors have made the corrections listed here.

Version 1.0.20 (September 15, 2018)

Changes

- The constraint \( a \neq 0 \) was added in (4.8.14).

Version 1.0.19 (June 22, 2018)

Equation (33.6.5)

\[
H^{\pm}_{\ell}(\eta, \rho) = \frac{e^{\pm i \theta_{\ell}(\eta, \rho)}}{(2\ell + 1)! \Gamma(-\ell \pm i \eta)} \left( \sum_{k=0}^{\infty} \frac{(a)_k}{(2\ell + 2)_k k!} (\mp 2i \rho)^{a+k} \left( \ln(\mp 2i \rho) + \psi(a+k) - \psi(1+k) - \psi(2\ell + 2 + k) \right) \right)
- \sum_{k=1}^{2\ell+1} \frac{(2\ell + 1)! (k-1)!}{(2\ell + 1-k)! (1-a)_k} (\mp 2i \rho)^{a-k}
\]

Originally the factor in the denominator on the right-hand side was written incorrectly as \( \Gamma(-\ell + i \eta) \). This has been corrected to \( \Gamma(-\ell \pm i \eta) \).

Reported by Ian Thompson on 2018-05-17

Sections 33.10(ii), 33.10(iii)

Originally it was stated incorrectly that \( \rho \) was fixed. This has been corrected to state that \( \eta \rho \) is fixed.

Reported by Ian Thompson on 2018-05-17

Equation (33.11.1)

\[
H^{\pm}_{\ell}(\eta, \rho) = e^{\pm i \theta_{\ell}(\eta, \rho)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (\pm 2i \rho)^k}
\]

Originally the factor in the denominator on the right-hand side was written incorrectly as \( (\mp 2i \rho)^k \). This has been corrected to \( (\pm 2i \rho)^k \).

Reported by Ian Thompson on 2018-05-17

- In Chapter 18, the reference Ismail (2005) has been replaced throughout by the further corrected paperback version Ismail (2009).
- In Section 36.1, the entry for \(*\) to represent complex conjugation was removed (see Version 1.0.19).
- In (36.2.18) and in §§36.12(i), 36.15(i), 36.15(ii), the vector at the origin, previously given as 0, has been clarified to read 0.
- A software bug that had corrupted some figures has been corrected.
Other Changes

- The overloaded operator ≡ is now more clearly separated (and linked) to two distinct cases: equivalence by definition (in §§1.4(ii), 1.4(v), 2.7(i), 2.10(iv), 3.1(i), 3.1(iv), 4.18, 9.18(ii), 9.18(vi), 18.2(iv), 20.2(iii), 20.7(vi), 23.20(ii), 25.10(i), 26.15, 31.17(i)); and modular equivalence (in §§24.10(i), 24.10(ii), 24.10(iii), 24.10(iv), 24.15(iii), 24.19(ii), 26.14(i), 26.21, 27.2(i), 27.8, 27.9, 27.11, 27.12, 27.14(v), 27.14(vi), 27.15, 27.16, 27.19).
- The notation and markup for complex conjugation has been made more consistent in §§1.17(iii), 9.9(i), 10.11, 10.34, 10.63(ii), 12.11(ii), 13.7(ii), 14.30(ii), 23.5(iv), 28.12(ii), 31.15(iii), 34.3(vii), 36.2(iii), 36.2(iv), 36.8, 36.11.
- In Chapter 35, the generalized hypergeometric function of matrix argument $pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; T)$, was labeled inadvertently as its single variable counterpart $pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; T)$. Furthermore, the Jacobi function of matrix argument $P_{\nu}^{(\gamma, \delta)}(\gamma, \delta)$, and the Laguerre function of matrix argument $L_{\nu}^{(\gamma)}(\gamma)$, were also labeled inadvertently (and incorrectly) in terms of the single variable counterparts given by $P_{\nu}^{(\gamma, \delta)}(\gamma, \delta)$, and $L_{\nu}^{(\gamma)}(\gamma)$. In order to resolve these inconsistencies, these functions now link correctly to their respective definitions.

Version 1.0.18 (March 27, 2018)

Table 5.4.1

The table of extrema for the Euler gamma function $\Gamma$ had several entries in the $x_n$ column that were wrong in the last 2 or 3 digits. These have been corrected and 10 extra decimal places have been included.

<table>
<thead>
<tr>
<th>n</th>
<th>$x_n$</th>
<th>$\Gamma(x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.46163 21449 68362 34126</td>
<td>0.88560 31944 10888 70028</td>
</tr>
<tr>
<td>1</td>
<td>-0.50408 30082 64455 40926</td>
<td>-3.54464 36111 55005 08912</td>
</tr>
<tr>
<td>2</td>
<td>-1.57349 84731 62390 45878</td>
<td>2.30240 72583 39680 13582</td>
</tr>
<tr>
<td>3</td>
<td>-2.61072 08684 44144 65000</td>
<td>-0.88813 63584 01241 92010</td>
</tr>
<tr>
<td>4</td>
<td>-3.63529 33664 36901 09784</td>
<td>0.24512 75398 34366 25044</td>
</tr>
<tr>
<td>5</td>
<td>-4.65323 77617 43142 44171</td>
<td>-0.05277 63935 87319 40076</td>
</tr>
<tr>
<td>6</td>
<td>-5.66716 24415 56885 53585</td>
<td>0.00932 45944 82614 85052</td>
</tr>
<tr>
<td>7</td>
<td>-6.67841 82130 73426 74283</td>
<td>-0.00139 73966 08949 76730</td>
</tr>
<tr>
<td>8</td>
<td>-7.68778 83250 31626 03744</td>
<td>0.00018 18784 44909 40419</td>
</tr>
<tr>
<td>9</td>
<td>-8.69576 41638 16401 26649</td>
<td>-0.00002 09252 90446 52667</td>
</tr>
<tr>
<td>10</td>
<td>-9.70267 25400 01863 73608</td>
<td>0.00000 21574 16104 52285</td>
</tr>
</tbody>
</table>

Reported 2018-02-17 by David Smith.

Other Changes

- The factor on the right-hand side of Equation (10.9.26) containing $\cos(\mu - \nu)\theta$ has been replaced with $\cos((\mu - \nu)\theta)$ to clarify the meaning.
- In Paragraph Confluent Hypergeometric Functions in §10.16, several Whittaker confluent hypergeometric functions were incorrectly linked to the definitions of the Kummer confluent hypergeometric and parabolic cylinder functions. However, to the eye, the functions appeared correct. The links were corrected.
• In Equation (15.6.9), it was clarified that $\lambda \in \mathbb{C}$.

• Originally Equation (19.16.9) had the constraint $a, a' > 0$. This constraint was replaced with $b_1 + \cdots + b_n \geq a > 0$, $b_j \in \mathbb{R}$. It therefore follows from Equation (19.16.10) that $a' > 0$. The last sentence of Subsection 19.16(ii) was elaborated to mention that generalizations may also be found in Carlson (1977). These were suggested by Bastien Roucariès.

• In Section 19.25(vi), the Weierstrass lattice roots $e_j$, were labeled inadvertently as the base of the natural logarithm. In order to resolve this inconsistency, the lattice roots $e_j$, and lattice invariants $g_2$, $g_3$, now link to their respective definitions (see §§23.2(i), 23.3(i)). This was reported by Felix Ospald.

• In Equation (19.25.37), the Weierstrass zeta function was incorrectly linked to the definition of the Riemann zeta function. However, to the eye, the function appeared correct. The link was corrected. 

• In Equation (27.12.5), the term originally written as $\sqrt{\ln x}$ was rewritten as $(\ln x)^{1/2}$ to be consistent with other equations in the same subsection.

**Version 1.0.17 (December 22, 2017)**

**Paragraph Mellin–Barnes Integrals in §8.6(ii)**

The descriptions for the paths of integration of the Mellin-Barnes integrals (8.6.10)–(8.6.12) have been updated. The description for (8.6.11) now states that the path of integration is to the right of all poles. Previously it stated incorrectly that the path of integration had to separate the poles of the gamma function from the pole at $s = 0$. The paths of integration for (8.6.10) and (8.6.12) have been clarified. In the case of (8.6.10), it separates the poles of the gamma function from the pole at $s = a$ for $\gamma(a, z)$. In the case of (8.6.12), it separates the poles of the gamma function from the poles at $s = 0, 1, 2, \ldots$.

*Reported 2017-07-10 by Kurt Fischer.*

**Section 10.37**

In §10.37, it was originally stated incorrectly that (10.37.1) holds for $|\operatorname{ph} z| < \pi$. The claim has been updated to $|\operatorname{ph} z| \leq \frac{1}{2} \pi$.

*Reported 2017-11-14 by Gergő Nemes.*

**Equation (10.37.1)**

$\Phi\left(\frac{\gamma(a, z)}{\phi(m)} + O\left(x \exp\left(-\lambda(\alpha)(\ln x)^{1/2}\right)\right),
\quad m \leq (\ln x)^{\alpha}, \quad \alpha > 0
\right)}$


• Bounds have been sharpened in §9.7(iii). The second paragraph now reads, “The nth error term is bounded in magnitude by the first neglected term multiplied by $\chi(n + \sigma) + 1$ where $\sigma = \frac{1}{8}$ for (9.7.7) and $\sigma = 0$ for (9.7.8), provided that $n \geq 0$ in the first case and $n \geq 1$ in the second case.” Previously it read, “In (9.7.7) and (9.7.8) the nth error term is bounded in magnitude by the first neglected term multiplied by $2\chi(n)\exp(\sigma\pi/(72\zeta))$ where $\sigma = 5$ for (9.7.7) and $\sigma = 7$ for (9.7.8), provided that $n \geq 1$ in both cases.” In Equation (9.7.16)

$$\text{Bi}(x) \leq \frac{e^x}{\sqrt{\pi x^{1/4}}} - \left(1 + \left(\frac{7}{4}\right) + 1\right) \frac{5}{72\zeta},$$

and (9.7.16)

$$\text{Bi}'(x) \leq \frac{x^{1/4}e^x}{\sqrt{\pi}} \left(1 + \left(\frac{\pi}{2} + 1\right) \frac{7}{72\zeta}\right),$$

the bounds on the right-hand sides have been sharpened. The factors $\left(\chi\left(\frac{5}{6}\right) + 1\right) \frac{5}{72\zeta}$, $\left(\frac{\pi}{2} + 1\right) \frac{7}{72\zeta}$, were originally given by $\frac{5\pi}{72\zeta}\exp\left(\frac{5\pi}{72\zeta}\right)$, $\frac{7\pi}{72\zeta}\exp\left(\frac{7\pi}{72\zeta}\right)$, respectively.

• Bounds have been sharpened in §9.7(iv). The first paragraph now reads, “The nth error term in (9.7.5) and (9.7.6) is bounded in magnitude by the first neglected term multiplied by

$$\left(1, \frac{1}{\frac{1}{3}\pi}, \text{ph}(z) \leq \frac{1}{3}\pi, \frac{1}{3}\pi \leq \text{ph}(z) \leq \frac{2}{3}\pi, \frac{2}{3}\pi \leq \text{ph}(z) < \pi, \min(|\csc(\phi z)|, \chi(n + \sigma) + 1), \chi(n + \sigma) + 1, \frac{3}{\pi} \leq \text{ph}(z) < \pi, \sqrt{2\pi n (\text{ph}(\phi z))} + \chi(n + \sigma) + 1, \chi(n + \sigma) + 1 \right).$$

provided that $n \geq 0$, $\sigma = \frac{1}{8}$ for (9.7.7) and $n \geq 1$, $\sigma = 0$ for (9.7.6).” Previously it read, “When $n \geq 1$ the nth error term in (9.7.5) and (9.7.6) is bounded in magnitude by the first neglected term multiplied by

$$2\exp\left(-\frac{\sigma}{36\zeta}\right) \left(1, \frac{1}{\frac{1}{3}\pi}, \frac{1}{\frac{1}{3}\pi} \leq \text{ph}(z) \leq \frac{2}{3}\pi, \frac{2}{3}\pi \leq \text{ph}(z) < \pi, \frac{2\chi(n)\exp(\sigma\pi/(72\zeta))}{\cos(\text{ph}(\phi z))^{n}} \left(1, \frac{1}{\frac{1}{3}\pi}, \frac{1}{\frac{1}{3}\pi} \leq \text{ph}(z) \leq \frac{2}{3}\pi, \frac{2}{3}\pi \leq \text{ph}(z) < \pi, \frac{4\chi(n)\exp(\sigma\pi)}{\cos(\text{ph}(\phi z))^{n}} \right).$$

Here $\sigma = 5$ for (9.7.5) and $\sigma = 7$ for (9.7.6).”

• In §10.8, a sentence was added just below (10.8.3) indicating that it is a rewriting of (16.12.1). This was suggested by Tom Koornwinder.

• Equations (10.15.1), (10.38.1), have been generalized to include the additional cases of $\partial J_{-\nu}(z)/\partial \nu$, $\partial I_{-\nu}(z)/\partial \nu$, respectively.

• The Kronecker delta symbols in Equations (10.22.37), (10.22.38), (14.17.6)–(14.17.9), have been moved furthest to the right, as is common convention for orthogonality relations.

• The titles of §§14.5(ii), 14.5(vi), have been changed to $\mu = 0$, $\nu = 0$, 1, and Addendum to §14.5(ii) : $\mu = 0$, $\nu = 2$, respectively, in order to be more descriptive of their contents.

• The second and the fourth lines of (19.7.2) containing $k'/k$ have both been replaced with $-ik'/k$ to clarify the meaning.

• Originally Equation (25.2.4) had the constraint $\Re s > 0$. This constraint was removed because, as stated after (25.2.1), $\zeta(s)$ is meromorphic with a simple pole at $s = 1$, and therefore $\zeta(s) - (s - 1)^{-1}$ is an entire function. This was suggested by John Harper.

• The title of §32.16 was changed from Physical to Physical Applications.

• Bibliographic citations and clarifications have been added, removed, or modified in §§5.6(i), 5.10, 7.8, 7.25(iii), and 32.16.

Version 1.0.16 (September 18, 2017)

Equation (8.12.18)

$$Q(a, z) = \frac{2\pi e^{-z}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{A_k(a)}{z^{k+2}} + \sum_{k=0}^{\infty} \frac{B_k(a)}{z^{k+2}}.

The original ± in front of the second summation was replaced by ¥ to correct an error in Paris (2002); for details see https://arxiv.org/abs/1611.00548.


Equation (14.5.14)

$$Q_{\nu}^{-1/2}(\cos \theta) = \left(\frac{\pi}{2 \sin \theta}\right)^{1/2} \cos\left((\nu + \frac{1}{2}) \theta\right)$$

Originally this equation was incorrect because of a minus sign in front of the right-hand side.

Reported 2017-04-10 by André Greiner-Petter.
Equations (17.2.22) and (17.2.23)

\[
17.2.22 \quad \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_{n}}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}; q)_{n}} = \frac{(aq^{2}; q^{2})_{n}}{(a; q^{2})_{n}} = \frac{1 - aq^{2n}}{1 - a} \\
17.2.23 \quad \frac{(qa^{\frac{1}{2}}, q\omega_{k}a^{\frac{1}{2}}, \ldots, q\omega_{k-1}a^{\frac{1}{2}}; q)_{n}}{(a^{\frac{1}{2}}, \omega_{k}a^{\frac{1}{2}}, \ldots, \omega_{k-1}a^{\frac{1}{2}}; q)_{n}} = \frac{(aq^{k}; q^{k})_{n}}{(a; q^{k})_{n}} = \frac{1 - aq^{kn}}{1 - a}
\]

The numerators of the leftmost fractions were corrected to read \((qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_{n}\) and \((qa^{\frac{1}{2}}, q\omega_{k}a^{\frac{1}{2}}, \ldots, q\omega_{k-1}a^{\frac{1}{2}}; q)_{n}\) instead of \((qa^{\frac{1}{2}}, -aq^{\frac{1}{2}}; q)_{n}\) and \((aq^{\frac{1}{2}}, q\omega_{k}a^{\frac{1}{2}}, \ldots, q\omega_{k-1}a^{\frac{1}{2}}; q)_{n}\), respectively.

Reported 2017-06-26 by Jason Zhao.

Equation (28.8.5)

\[
28.8.5 \quad V_{m}(\xi) \sim \frac{1}{2\sqrt{\hbar}} \left(-D_{m+2}(\xi) - m(m - 1)D_{m-2}(\xi)\right) + \frac{1}{2!}\hbar^{2} \left(D_{m+6}(\xi) + (m^{2} - 25m - 36)D_{m+2}(\xi) - m(m - 1)(m^{2} + 27m - 10)D_{m-2}(\xi) - 6!(m_{6})D_{m-6}(\xi)\right) + \cdots
\]

Originally the – in front of the 6! was given incorrectly as +.

Reported 2017-02-02 by Daniel Karlsson.

Other Changes

- To be consistent with the notation used in (8.12.16), Equation (8.12.5) was changed to read

\[
8.12.5 \quad \frac{e^{\pm i\alpha}}{2i\sin(\pi a)} Q(-a, z e^{\pm i\alpha}) = \pm \frac{1}{2} \text{erfc} \left(\pm i\eta\sqrt{a/2}\right) - iT(a, \eta)
\]

- Following a suggestion from James McTavish on 2017-04-06, the recurrence relation \(u_{k} = \frac{(6k - 5)(6k - 3)(6k - 1)}{(2k - 1)(216k)} u_{k-1}\) was added to Equation (9.7.2).

- In §15.2(ii), the unnumbered equation

\[
\lim_{c \to -n} \frac{F(a, b; c; z)}{\Gamma(c)} = F(a, b; -n; z) = \frac{(a)_{n+1}(b)_{n+1} z^{n+1} F(a + n + 1, b + n + 1; n + 2; z)}{(n + 1)!}
\]

was added in the second paragraph. An equation number will be assigned in an expanded numbering scheme that is under current development. Additionally, the discussion following (15.2.6) was expanded.

- In §15.4(i), due to a report by Louis Klauder on 2017-01-01, and in §15.4(iii), sentences were added specifying that some equations in these subsections require special care under certain circumstances. Also, (15.4.6) was expanded by adding the formula \(F(a, b; a; z) = (1 - z)^{-b}\).
A bibliographic citation was added in §11.13(i).

Version 1.0.15 (June 1, 2017)

Changes

- There have been extensive changes in the notation used for the integral transforms defined in §1.14. These changes are applied throughout the DLMF. The following table summarizes the changes.

<table>
<thead>
<tr>
<th>Transform</th>
<th>New Notation</th>
<th>Abbreviated Notation</th>
<th>Old Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fourier</td>
<td>$\mathcal{F}(f)(x)$</td>
<td>$\mathcal{F}f(x)$</td>
<td>$\mathcal{F}f(x)$</td>
</tr>
<tr>
<td>Fourier Cosine</td>
<td>$\mathcal{F}_c(f)(x)$</td>
<td>$\mathcal{F}_c f(x)$</td>
<td>$\mathcal{F}_c f(x)$</td>
</tr>
<tr>
<td>Fourier Sine</td>
<td>$\mathcal{F}_s(f)(x)$</td>
<td>$\mathcal{F}_s f(x)$</td>
<td>$\mathcal{F}_s f(x)$</td>
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<td>$\mathcal{L}f(x)$</td>
<td>$\mathcal{L}f(x)$</td>
</tr>
<tr>
<td>Mellin</td>
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<td>$\mathcal{M}f(s)$</td>
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<tr>
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<td>$\mathcal{H}f(s)$</td>
<td>$\mathcal{H}f(s)$</td>
</tr>
<tr>
<td>Stieltjes</td>
<td>$\mathcal{S}(f)(s)$</td>
<td>$\mathcal{S}f(s)$</td>
<td>$\mathcal{S}f(s)$</td>
</tr>
</tbody>
</table>

Previously, for the Fourier, Fourier cosine and Fourier sine transforms, either temporary local notations were used or the Fourier integrals were written out explicitly.

- Several changes have been made in §1.16(vii) to
  (i) make consistent use of the Fourier transform notations $\mathcal{F}(f)$, $\mathcal{F}(\phi)$ and $\mathcal{F}(u)$ where $f$ is a function of one real variable, $\phi$ is a test function of $n$ variables associated with tempered distributions, and $u$ is a tempered distribution (see (1.14.1), (1.16.29) and (1.16.35));
  (ii) introduce the partial differential operator $\mathbf{D}$ in (1.16.30);
  (iii) clarify the definition (1.16.32) of the partial differential operator $P(\mathbf{D})$; and
  (iv) clarify the use of $P(\mathbf{D})$ and $P(x)$ in (1.16.33), (1.16.34), (1.16.36) and (1.16.37).

- An entire new Subsection 1.16(viii) Fourier Transforms of Special Distributions, was contributed by Roderick Wong.

- The validity constraint $|\text{ph} z| < \frac{1}{2}\pi$ was added to (9.5.6). Additionally, specific source citations are now given in the metadata for all equations in Chapter 9.

- The relation between Clebsch-Gordan and 3j symbols was clarified, and the sign of $m_3$ was changed for readability. The reference Condon and Shortley (1935) for the Clebsch-Gordan coefficients was replaced by Edmonds (1974) and Rotenberg et al. (1959) and the references for 3j, 6j, 9j symbols were made more precise in §34.1.

- The website’s icons and graphical decorations were upgraded to use SVG, and additional icons and mouse-cursors were employed to improve usability of the interactive figures.

Version 1.0.14 (December 21, 2016)

Equation (8.18.3)

8.18.3 $I_x(a, b) = \frac{\Gamma(a + b)}{\Gamma(a)} \left( \sum_{k=0}^{n-1} d_k F_k + O(a^{-n}) F_0 \right)$

The range of $x$ was extended to include 1. Previously this equation appeared without the order estimate as $I_x(a, b) \sim \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{k=0}^{\infty} d_k F_k$.

Reported 2016-08-30 by Xinrong Ma.

Equation (17.9.2)

17.9.2 $\begin{array}{l}
2\phi_1\left( q^{-n}, b;c ; q, z \right) \\
= \frac{(c/b;q)_n}{(c;q)_n} n^{-3} \phi_1 \left( q^{-n}, b, q/z; b q^{1-n}/c ; q, z/c \right)
\end{array}$

The entry $q/c$ in the first row of $3\phi_1\left( q^{-n}, b q/c; q, z/c \right)$ was replaced by $q/z$.

Reported 2016-08-30 by Xinrong Ma.

Figures 36.3.9, 36.3.10, 36.3.11, 36.3.12

Scales were corrected in all figures. The interval $-8.4 \leq \frac{x+y}{\sqrt{2}} \leq 8.4$ was replaced by $-12.0 \leq \frac{x+y}{\sqrt{2}} \leq 12.0$ and $-12.7 \leq \frac{x+y}{\sqrt{2}} \leq 12.7$ replaced by $-18.0 \leq \frac{x+y}{\sqrt{2}} \leq 6.0$. All plots and interactive visualizations were regenerated to improve image quality.
Figure 36.3.9: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 0)|$.

Figure 36.3.10: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 1)|$.

Figure 36.3.11: Modulus of hyperbolic umbilic canonical integral function $|\Psi^{(H)}(x, y, 2)|$. 
Figures 36.3.12, 36.3.18, 36.3.19, 36.3.20, 36.3.21

The scaling error reported on 2016-09-12 by Dan Piponi also applied to contour and density plots for the phase of the hyperbolic umbilic canonical integrals. Scales were corrected in all figures. The interval $-8.4 \leq \frac{x-y}{\sqrt{2}} \leq 8.4$ was replaced by $-12.0 \leq \frac{x-y}{\sqrt{2}} \leq 12.0$ and $-12.7 \leq \frac{x+y}{\sqrt{2}} \leq 4.2$ replaced by $-18.0 \leq \frac{x+y}{\sqrt{2}} \leq 6.0$. All plots and interactive visualizations were regenerated to improve image quality.
Figure 36.3.19: Phase of hyperbolic umbilic canonical integral $\Psi^{(H)}(x, y, 1)$.

Figure 36.3.20: Phase of hyperbolic umbilic canonical integral $\Psi^{(H)}(x, y, 2)$.

Figure 36.3.21: Phase of hyperbolic umbilic canonical integral $\Psi^{(H)}(x, y, 3)$.

Reported 2016-09-28.
Other Changes

- A number of changes were made with regard to fractional integrals and derivatives. In §1.15(vi) a reference to Miller and Ross (1993) was added, the fractional integral operator of order $\alpha$ was more precisely identified as the Riemann-Liouville fractional integral operator of order $\alpha$, and a paragraph was added below (1.15.50) to generalize (1.15.47). In §1.15(vii) the sentence defining the fractional derivative was clarified. In §2.6(iii) the identification of the Riemann-Liouville fractional integral operator was made consistent with §1.15(vi).

- Changes to §8.18(ii)–§8.11(v): A sentence was added in §8.18(ii) to refer to Nemes and Olde Daalhuis (2016). Originally §8.11(iii) was applicable for real variables $a$ and $x = \lambda a$. It has been extended to allow for complex variables $a$ and $z = \lambda a$ (and we have replaced $x$ with $z$ in the subsection heading and in Equations (8.11.6) and (8.11.7)). Also, we have added two paragraphs after (8.11.9) to replace the original paragraph that appeared there. Furthermore, the interval of validity of (8.11.6) was increased from $0 < \lambda < 1$ to the sector $0 < \lambda < 1$, $|\text{ph} a| \leq \frac{\pi}{2} - \delta$, and the interval of validity of (8.11.7) was increased from $\lambda > 1$ to the sector $\lambda > 1$, $|\text{ph} a| \leq \frac{3\pi}{2} - \delta$. A paragraph with reference to Nemes (2016) has been added in §8.11(v), and the sector of validity for (8.11.12) was increased from $|\text{ph} z| \leq \pi - \delta$ to $|\text{ph} z| \leq 2\pi - \delta$. Two new Subsections 13.6(vii), 13.18(vi), both entitled Coulomb Functions, were added to note the relationship of the Kummer and Whittaker functions to various forms of the Coulomb functions. A sentence was added in both §13.10(vi) and §13.23(v) noting that certain generalized orthogonality can be expressed in terms of Kummer functions.

- Four of the terms in (14.15.23) were rewritten for improved clarity.

- In §15.6 it was noted that (15.6.8) can be rewritten as a fractional integral.

- In applying changes in Version 1.0.12 to (16.15.3), an editing error was made; it has been corrected.

- In §34.1, the reference for Clebsch-Gordan coefficients, Condon and Shortley (1935), was replaced by Edmonds (1974) and Rothenberg et al. (1959). The references for $3j$, $6j$, $9j$ symbols were made more precise.

- Images in Figures 36.3.1, 36.3.2, 36.3.3, 36.3.4, 36.3.5, 36.3.6, 36.3.7, 36.3.8 and Figures 36.3.13, 36.3.14, 36.3.15, 36.3.16, 36.3.17 were resized for consistency.

- Meta.Numerics (website) was added to the Software Table at http://dlmf.nist.gov/software/.

Version 1.0.13 (September 16, 2016)

Other Changes

- In applying changes in Version 1.0.12 to (13.9.16), an editing error was made; it has been corrected.

Version 1.0.12 (September 9, 2016)

Equations (25.11.6), (25.11.19), and (25.11.20)

Originally all six integrands in these equations were incorrect because their numerators contained the function $B_2(x)$. The correct function is $\frac{B_2(x) - B_2}{2}$. The new equations are:

\begin{equation}
25.11.6 \quad \zeta(s, a) = \frac{1}{a^s} \left( \frac{1}{2} + \frac{a}{s - 1} \right) - \frac{s(s + 1)}{2} \int_0^\infty \frac{B_2(x) - B_2}{(x + a)^{s+2}} \, dx, \quad s \neq 1, \Re s > -1, a > 0
\end{equation}

Reported 2016-05-08 by Clemens Heuberger.

\begin{equation}
25.11.19 \quad \zeta'(s, a) = -\frac{\ln a}{a^s} \left( \frac{1}{2} + \frac{a}{s - 1} \right) - \frac{a^{-1-s}}{(s-1)^2} + \frac{s(s+1)}{2} \int_0^\infty \frac{\tilde{B}_2(x) - B_2}{(x + a)^{s+2}} \ln(x + a) \, dx
\end{equation}
Reported 2016-06-27 by Gergő Nemes.

25.11.20

\[ (-1)^k \zeta^{(k)}(s, a) = \frac{(\ln a)^k}{a^s} \left( \frac{1}{2} + \frac{a}{s-1} \right) + k! a^{1-s} \sum_{r=0}^{k-1} \frac{(\ln a)^r}{r!(s-1)^{k-r+1}} \]

\[- \frac{s(s+1)}{2} \int_0^\infty \frac{(\bar{B}_2(x) - B_2)(\ln(x+a))^k}{(x+a)^{s+2}} \, dx + \frac{k(2s+1)}{2} \int_0^\infty \frac{(\bar{B}_2(x) - B_2)(\ln(x+a))^{k-1}}{(x+a)^{s+2}} \, dx \]

\[- \frac{k(k-1)}{2} \int_0^\infty \frac{(\bar{B}_2(x) - B_2)(\ln(x+a))^{k-2}}{(x+a)^{s+2}} \, dx, \quad \Re s > -1, \ s \neq 1, \ a > 0 \]

Reported 2016-06-27 by Gergő Nemes.

Other Changes

- The symbol \( \sim \) is used for two purposes in the DLMF, in some cases for asymptotic equality and in other cases for asymptotic expansion, but links to the appropriate definitions were not provided. In this release changes have been made to provide these links.

- A short paragraph dealing with asymptotic approximations that are expressed in terms of two or more Poincaré asymptotic expansions has been added in §2.1(iii) below (2.1.16).

- Because (2.11.4) is not an asymptotic expansion, the symbol \( \sim \) that was used originally is incorrect and has been replaced with \( \approx \), together with a slight change of wording.

- Originally (13.9.16) was expressed in term of asymptotic symbol \( \sim \). As a consequence of the use of the \( O \) order symbol on the right hand side, \( \sim \) was replaced by \( = \).

- In (13.2.9) and (13.2.10) there were clarifications made in the conditions on the parameter \( a \) in \( U(a, b, z) \) of those equations.

- Originally (14.15.23) used \( f(x) \) to represent both \( U(-c, x) \) and \( \overline{U}(-c, x) \). This has been replaced by two equations giving explicit definitions for the two envelope functions. Some slight changes in wording were needed to make this clear to readers.

- The title for §17.9 was changed from Transformations of Higher \( \phi \), Functions to Further Transformations of \( r+1 \phi \), Functions.

- A number of additions and changes have been made to the metadata in Chapter 25 to reflect new and changed references as well as to how some equations have been derived.

- Bibliographic citations, clarifications, typographical corrections and added or modified sentences appear in §§18.15(i) and 18.16(ii).

Version 1.0.11 (June 8, 2016)

Figure 4.3.1

This figure was rescaled, with symmetry lines added, to make evident the symmetry due to the inverse relationship between the two functions.

Equation (9.7.17)

Originally the constraint, \( \frac{3}{4} \pi \leq |\text{ph} \, z| < \pi \), was written incorrectly as \( \frac{3}{4} \pi \leq |\text{ph} \, z| \leq \pi \). Also, the equation was reformatted to display the constraints in the equation instead of in the text.

Reported 2014-11-05 by Gergő Nemes.
Equation (10.32.13)

Originally the constraint, $|\phi z| < \frac{1}{2} \pi$, was incorrectly written as, $|\phi z| < \pi$.


Equation (10.40.12)

Originally the third constraint $\pi \leq |\phi z| < \frac{3}{2} \pi$ was incorrectly written as $\pi \leq |\phi z| \leq \frac{3}{2} \pi$.

Reported 2014-11-05 by Gergő Nemes.

Equation (23.18.7)

23.18.7 $s(d, c) = \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \frac{1}{2} \right)$, $c > 0$

Originally the sum $\sum_{r=1}^{c-1}$ was written with an additional condition on the summation, that $(r, c) = 1$. This additional condition was incorrect and has been removed.

Reported 2015-10-05 by Howard Cohl and Tanay Wakhare.

Equations (28.28.21) and (28.28.22)

28.28.21

\[
\frac{4}{\pi} \int_0^{\pi/2} C_{2\ell+1}(2hR) \cos((2\ell + 1)\phi) c_{2m+1}(t, h^2) \, dt = (-1)^{\ell+m} A_{2m+1}(h^2) M_{2m+1}(z, h)
\]

28.28.22

\[
\frac{4}{\pi} \int_0^{\pi/2} C_{2\ell+1}(2hR) \sin((2\ell + 1)\phi) c_{2m+1}(t, h^2) \, dt = (-1)^{\ell+m} B_{2\ell+1}(h^2) M_{2m+1}(z, h),
\]

Originally the prefactor $\frac{4}{\pi}$ and upper limit of integration $\pi/2$ in these two equations were given incorrectly as $\frac{2}{\pi}$ and $\pi$.

Reported 2015-05-20 by Ruslan Kabasayev

\begin{itemize}
  \item It was reported by Nico Temme on 2015-02-28 that the asymptotic formula for $\ln \Gamma(z + h)$ given in (5.11.8) is valid for $h \in \mathbb{C}$; originally it was unnecessarily restricted to $[0,1]$.
  \item In §13.8(iii), a new paragraph with several new equations and a new reference has been added at the end to provide asymptotic expansions for Kummer functions $U(a, b, z)$ and $M(a, b, z)$ as $a \to \infty$ in $|\phi a| \leq \pi - \delta$ and $b$ and $z$ fixed.
  \item Because of the use of the $O$ order symbol on the right-hand side, the asymptotic expansion (18.15.22) for the generalized Laguerre polynomial $L_n^{(\alpha)}(\nu x)$ was rewritten as an equality.
  \item The entire Section 27.20 was replaced.
  \item Bibliographic citations have been added or modified in §§2.4(v), 2.4(vi), 2.9(iii), 5.11(i), 5.11(ii), 5.17, 9.9(i), 10.22(v), 10.37, 11.6(iii), 11.9(iii), 12.9(i), 13.8(ii), 13.11, 14.15(i), 14.15(ii), 15.12(iii), 15.14, 16.11(ii), 16.13, 18.15(vi), 20.7(viii), 24.11, 24.16(i), 26.8(vii), 33.12(i), and 33.12(ii).
  \item Clarifications, typographic corrections, added or modified sentences appear in §§1.2(i), 2.4(i), 2.9(iii), 4.4.2(i), 11.11.1, 4.4.2(i), 11.11.9, 21.5.7, and 27.14.7.
\end{itemize}

Version 1.0.10 (August 7, 2015)

Section 4.43

The first paragraph has been rewritten to correct reported errors. The new version is reproduced here.

Let $p (\neq 0)$ and $q$ be real constants and

4.43.1 $A = \left(-\frac{4}{3}p\right)^{1/2}, \quad B = \left(\frac{4}{3}p\right)^{1/2}$.

The roots of

4.43.2 $z^3 + pz + q = 0$

are:

(a) $A \sin a$, $A \sin(a + \frac{2}{3} \pi)$, and $A \sin(a + \frac{4}{3} \pi)$, with $\sin(3a) = 4q/A^3$, when $4p^3 + 27q^2 \leq 0$.

(b) $A \cosh a$, $A \cosh(a + \frac{2}{3} \pi)$, and $A \cosh(a + \frac{4}{3} \pi)$, with $\cosh(3a) = -4q/A^3$, when $p < 0$, $q < 0$, and $4p^3 + 27q^2 > 0$.

(c) $B \sinh a$, $B \sinh(a + \frac{2}{3} \pi)$, and $B \sinh(a + \frac{4}{3} \pi)$, with $\sinh(3a) = -4q/B^3$, when $p > 0$. 

\begin{itemize}
  \item In §1.2(i), a sentence was added after (1.2.1) to refer to (1.2.6) as the definition of the binomial coefficient $\binom{z}{k}$ when $z$ is complex. As a notational clarification, wherever $n$ appeared originally in (1.2.6)–(1.2.9), it was replaced by $z$.
\end{itemize}
Note that in Case (a) all the roots are real, whereas in Cases (b) and (c) there is one real root and a conjugate pair of complex roots. See also §1.11(iii).

Reported 2014-10-31 by Masataka Urago.

Equation (9.10.18)

\[ \text{Ai}(z) = \frac{3z^{5/4}e^{-(2/3)z^{3/2}}}{4\pi} \int_0^\infty t^{-3/4}e^{-(2/3)t^{3/2}}Ai(t) \, dt \]

9.10.18

The original equation taken from Schulten et al. (1979) was incorrect.

Reported 2015-03-20 by Walter Gautschi.

Equation (9.10.19)

\[ \text{Bi}(x) = \frac{3x^{5/4}e^{(2/3)x^3/2}}{2\pi} \int_0^\infty t^{-3/4}e^{-(2/3)t^{3/2}}Ai(t) \, dt \]

9.10.19

The original equation taken from Schulten et al. (1979) was incorrect.

Reported 2015-03-20 by Walter Gautschi.

Equation (10.17.14)

\[ |R^\ell_\nu(z)| \leq 2|a_\nu(n)|V_{z,\pm i\infty}(t^{-1}) \times \exp\left(|\nu^2 - \frac{1}{4}|V_{z,\pm i\infty}(t^{-1})\right) \]

10.17.14

Originally the factor \( V_{z,\pm i\infty}(t^{-1}) \) in the argument to the exponential was written incorrectly as \( V_{z,\pm i\infty}(t^{-1}) \).

Reported 2014-09-27 by Gergő Nemes.

Equation (10.19.11)

\[ Q_3(a) = \frac{549}{28000}a^8 - \frac{19076}{63000}a^5 + \frac{79}{2376}a^2 \]

10.19.11

Originally the first term on the right-hand side of this equation was written incorrectly as \(-\frac{549}{28000}a^8\).

Reported 2015-03-16 by Svante Janson.

Equation (12.3.27)

\[ U(-m, b, z) = (-1)^m(b)_mM(-m, b, z) \]

12.3.27

\[ = (-1)^m \sum_{s=0}^m \binom{m}{s}(b+s)_{m-s}(-z)^s \]

The equality \( U(-m, b, z) = (-1)^m(b)_mM(-m, b, z) \) has been added to the original equation to express an explicit connection between the two standard solutions of Kummer’s equation. Note also that the notation \( a = -n \) has been changed to \( a = -m \).

Reported 2015-02-10 by Adri Olde Daalhuis.

Equation (13.2.8)

\[ U(a, a+n+1, z) = \frac{(-1)^n(1-a-n)z^{\alpha+n}}{\pi^{3/2}M(-n, 1-a-n, z)} \]

13.2.8

\[ \times M(-n, 1-a-n, z) \]

\[ = z^{-\alpha} \sum_{s=0}^n \binom{n}{s}(a)_s z^{-s} \]

The equality \( U(a, a+n+1, z) = \frac{(-1)^n(1-a-n)z^{\alpha+n}}{\pi^{3/2}} \times M(-n, 1-a-n, z) \) has been added to the original equation to express an explicit connection between the two standard solutions of Kummer’s equation.

Reported 2015-02-10 by Adri Olde Daalhuis.

Equation (13.2.10)

\[ U(-m, n+1, z) \]

13.2.10

\[ = (-1)^m(n+1)_mM(-m, n+1, z) \]

\[ = (-1)^m \sum_{s=0}^m \binom{m}{s}(n+s+1)_{m-s}(-z)^s \]

The equality \( U(-m, n+1, z) = (-1)^m(n+1)_m \times M(-m, n+1, z) \) has been added to the original equation to express an explicit connection between the two standard solutions of Kummer’s equation. Note also that the notation \( a = -m, m = 0, 1, 2, \ldots \) has been introduced.

Reported 2015-02-10 by Adri Olde Daalhuis.

Equation (18.33.3)

\[ \phi_\alpha^n(z) = z^n\phi_\alpha(z) = \kappa_n + \sum_{\ell=1}^n \kappa_{n, n-\ell}z^\ell \]

18.33.3

\[ \phi_\alpha^n(z) = z^n\phi_\alpha(z) = \kappa_n + \sum_{\ell=1}^n \kappa_{n, n-\ell}z^\ell \]

Originally this equation was written incorrectly as \( \phi_\alpha^n(z) = \kappa_n z^n + \sum_{\ell=1}^n \kappa_{n, n-\ell}z^{n-\ell} \). Also, the equality \( \phi_\alpha^n(z) = z^n\phi_\alpha(z) \) has been added.

Reported 2014-10-03 by Roderick Wong.
Equation (34.7.4)

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  m_1 & m_2 & m_3
\end{pmatrix}
= \sum_{m_{r1}, m_{r2}, r=1,2,3}
\begin{pmatrix}
  j_{11} & j_{12} & j_{13} \\
  m_{11} & m_{12} & m_{13}
\end{pmatrix}
\times \begin{pmatrix}
  j_{21} & j_{22} & j_{23} \\
  m_{21} & m_{22} & m_{23}
\end{pmatrix}
\times \begin{pmatrix}
  j_{31} & j_{32} & j_{33} \\
  m_{31} & m_{32} & m_{33}
\end{pmatrix}
\]

Originally the third $3j$ symbol in the summation was written incorrectly as \( \begin{pmatrix} j_{13} & j_{23} & j_{33} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} \).

Reported 2015-01-19 by Yan-Rui Liu.

Other Changes

- To increase the regions of validity (5.9.10), (5.9.11), (5.10.1), (5.11.1), and (5.11.8), the logarithms of the gamma function that appears on their left-hand sides have all been changed to \( \ln \Gamma (\cdot) \), where \( \ln \) is the general logarithm. Originally \( \ln \Gamma (\cdot) \) was used, where \( \ln \) is the principal branch of the logarithm. These changes were recommended by Philippe Spindel on 2015-02-06.

- The notation used for the \( q \)-Appell functions in Section 17.1 and Equations (17.4.5), (17.4.6), (17.4.7), (17.4.8), (17.11.1), (17.11.2) and (17.11.3) was updated to explicitly include the argument \( q \), as used in Gasper and Rahman (2004).

- A note was added after (22.20.5) to deal with cases when computation of \( du(x, k) \) becomes numerically unstable near \( x = K \).

- The spelling of the name Delannoy was corrected in several places in §26.6. Previously it was misspelled as Dellanoy.

- For consistency of notation across all chapters, the notation for logarithm has been changed to \( \ln \) from \( \log \) throughout Chapter 27.

Equation (34.3.7)

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_1 & -j_1 - m_3 & m_3
\end{pmatrix}
= (-1)^{j_1-j_2-m_3}
\left( \frac{(2j_1)!(j_1+j_2+j_3)!(j_1+j_2+m_3)!(j_3-m_3)!}{(j_1+j_2+j_3+1)!(j_1-j_2+j_3)!(j_1+j_2-j_3)!(-j_1+j_2-m_3)!(j_3+m_3)!} \right)^{1/2}
\]

- Bibliographic citations have been added or modified in §§2.4(vi), 3.8(v), 5.6(i), 5.10, 5.11(i), 5.11(ii), 5.18(ii), 7.21, 8.10, 10.21(ix), 10.45, 10.74(vi), 11.7(v), 13.7(iii), 14.17(iii), 14.20(ix), 14.28(ii), 14.32, 15.8(v), 15.13, 15.19(i), 16.6, 16.13, 17.6(ii), 17.7(iii), 18.1(iii), 18.3, 18.15(iv) and 18.24.

**Version 1.0.9 (August 29, 2014)**

Equation (9.6.26)

\[
\text{Bi}'(z) = \frac{3^{1/6}}{\Gamma(\frac{1}{3})} e^{-\zeta} F_1 \left( -\frac{1}{6}; -\frac{1}{3}; 2\zeta \right) + \frac{3^{7/6}}{2^{7/3} \Gamma(\frac{1}{3})} \zeta^{4/3} e^{-\zeta} F_1 \left( \frac{7}{6}; \frac{7}{3}; 2\zeta \right)
\]

Originally the second occurrence of the function \( F_1 \) was given incorrectly as \( F_1 \left( \frac{7}{6}; \frac{7}{3}; \zeta \right) \).

Reported 2014-05-21 by Hanyou Chu.

Equation (22.19.6)

\[
x(t) = \text{cn} \left( \sqrt{1 + 2\eta} k \right)
\]

Originally the term \( \sqrt{1 + 2\eta} \) was given incorrectly as \( \sqrt{1 + \eta} \) in this equation and in the line above. Additionally, for improved clarity, the modulus \( k = 1/\sqrt{2 + \eta^{-1}} \) has been defined in the line above.

Reported 2014-05-02 by Svante Janson.

**Paragraph Case III:** \( V(x) = -\frac{1}{2}x^2 + \frac{1}{4}\beta x^4 \) in §22.19(ii)

Two corrections have been made in this paragraph. First, the correct range of the initial displacement \( a \) is \( \sqrt{1/\beta} \leq |a| < \sqrt{2/\beta} \). Previously it was \( \sqrt{1/\beta} \leq |a| \leq \sqrt{2/\beta} \). Second, the correct period of the oscillations is \( 2K(k) / \sqrt{\eta} \). Previously it was given incorrectly as \( 4K(k) / \sqrt{\eta} \).

Reported 2014-05-02 by Svante Janson.
In the original equation the prefactor of the above 3j symbol read \((-1)^{j_1-j_2-j_3-m_3}\). It is now replaced by its correct value \((-1)^{j_1-j_2+m_3}\).

Reported 2014-06-12 by James Zibin.

Other Changes

- Pochhammer symbols have been introduced in Equations (7.12.1), (7.12.2), (7.12.3), (7.12.4), (7.12.5), (25.5.7), (25.8.3), (25.11.10), (25.11.28), and (25.11.43) to make the notation more concise.
- The Wronskian (14.2.7) was generalized to include both associated Legendre and Ferrers functions.
- A cross-reference has been added in §15.9(iv).
- Equations (22.19.6), (22.19.7), (22.19.8), and (22.19.9) have been rewritten with the modulus (second argument) of the Jacobian elliptic function defined explicitly in the preceding line of text.
- Bibliographic citations have been added in §§4.13, 4.48(iv), 6.21(ii), 8.28(ii), 9.16, 10.77(viii), 12.21(ii), 14.28(ii), 14.34(ii), 16.4(ii) and 16.13.
- An addition was made to the Software Table at http://dlmf.nist.gov/software/ to reflect the addition of a multiple precision (MP) package written in C++ which uses a variety of different MP interfaces.

Version 1.0.8 (April 25, 2014)

Equation (22.19.2)

22.19.2 \[ \sin\left(\frac{1}{2} \theta(t)\right) = \sin\left(\frac{1}{2} \alpha\right) \sin(t + K, \sin\left(\frac{1}{2} \alpha\right)) \]

Originally the first argument to the function \(\sin\) was given incorrectly as \(t\). The correct argument is \(t + K\).

Reported 2014-03-05 by Svante Janson.

Equation (22.19.3)

22.19.3 \[ \theta(t) = 2 \text{am}\left(t \sqrt{E}/2, \sqrt{2}/E\right) \]

Originally the first argument to the function \(\text{am}\) was given incorrectly as \(t\). The correct argument is \(t \sqrt{E}/2\).

Reported 2014-03-05 by Svante Janson.

Other Changes

- Minor additions have been made in §§9.6(iii), 22.19(i).
- Equation (10.13.4) has been generalized to cover an additional case.
- We avoid the troublesome symbols, often missing in installed fonts, previously used for exponential e, imaginary i and differential d.

Version 1.0.7 (March 21, 2014)

Table 3.5.19

The correct headings for the second and third columns of this table are \(J_0(t)\) and \(g(t)\), respectively. Previously these columns were mislabeled as \(g(t)\) and \(J_0(t)\).

<table>
<thead>
<tr>
<th>(t)</th>
<th>(J_0(t))</th>
<th>(g(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000 00000</td>
<td>1.00000 00000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.93846 98072</td>
<td>0.93846 98072</td>
</tr>
<tr>
<td>1.0</td>
<td>0.76519 76866</td>
<td>0.76519 76865</td>
</tr>
<tr>
<td>2.0</td>
<td>0.22389 07791</td>
<td>0.22389 10326</td>
</tr>
<tr>
<td>5.0</td>
<td>-0.17759 67713</td>
<td>-0.17902 54097</td>
</tr>
<tr>
<td>10.0</td>
<td>-0.24593 57645</td>
<td>-0.07540 53543</td>
</tr>
</tbody>
</table>

Reported 2014-01-31 by Masataka Urago.

Table 3.5.21

The correct corner coordinates for the 9-point square, given on the last line of this table, are \((\pm \sqrt{\frac{5}{8}} h, \pm \sqrt{\frac{5}{8}} h)\). Originally they were given incorrectly as \((\pm \sqrt{\frac{5}{8}} h, 0), (\pm \sqrt{\frac{5}{8}} h, 0)\).
Equation (4.21.1)

\[
\sin u \pm \cos u = \sqrt{2} \sin \left( u \pm \frac{1}{4} \pi \right)
\]

\[
= \pm \sqrt{2} \cos \left( u \mp \frac{1}{4} \pi \right)
\]

Originally the symbol \(\pm\) was missing after the second equal sign.

---

Equation (4.23.34) and (4.23.35)

4.23.34 \[ \arcsin z = \arcsin \beta + i \text{sign}(y) \ln \left( \alpha + (\alpha^2 - 1)^{1/2} \right) \]

and

4.23.35 \[ \arccos z = \arccos \beta - i \text{sign}(y) \ln \left( \alpha + (\alpha^2 - 1)^{1/2} \right) \]

Originally the factor sign(y) was missing from the second term on the right sides of these equations. Additionally, the condition for the validity of these equations has been weakened.

Reported 2013-07-01 by Volker Thürey.

Equation (5.17.5)

\[
\ln G(z + 1) \sim \frac{1}{2} z^2 + z \ln \Gamma(z + 1) - \left( \frac{1}{2} z(z + 1) + \frac{1}{12} \right) \ln z
\]

\[
- \ln A + \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}}
\]

Originally the term \(z \ln \Gamma(z + 1)\) was incorrectly stated as \(2\Gamma(z + 1)\).

Reported 2013-08-01 by Gergő Nemes and subsequently by Nick Jones on December 11, 2013.

Table 22.4.3

A correction was made in the online portion of this table.

Reported 2014-02-28 by Svante Janson.

Table 22.5.2

The entry for \(\text{sn} z \) at \( z = \frac{3}{2}(K + iK') \) has been corrected. The correct entry is \((1+i)((1+k')^{1/2} - i(1-k')^{1/2})/(2k^{1/2})\). Originally the terms \((1+k')^{1/2}\) and \((1-k')^{1/2}\) were given incorrectly as \((1+k')^{1/2}\) and \((1-k')^{1/2}\).

Similarly, the entry for \(\text{dn} z \) at \( z = \frac{3}{2}(K + iK') \) has been corrected. The correct entry is \((-1+i)k'^{1/2}((1+k')^{1/2} + i(1-k')^{1/2})/2\). Originally the terms \((1+k')^{1/2}\) and \((1-k')^{1/2}\) were given incorrectly as \((1+k')^{1/2}\) and \((1-k')^{1/2}\).

Reported 2014-02-28 by Svante Janson.
Equation (22.6.7)

\[ dn(2z, k) = \frac{dn^2(z, k) - k^2 sn^2(z, k) cn^2(z, k)}{1 - k^2 sn^4(z, k)} \]

22.6.7

\[ = \frac{dn^4(z, k) + k^2 k'^2 sn^4(z, k)}{1 - k^2 sn^4(z, k)} \]

Originally the term \( k^2 sn^2(z, k) cn^2(z, k) \) was given incorrectly as \( k^2 sn^2(z, k) dn^2(z, k) \).

Reported 2014-02-28 by Svante Janson.

Table 26.8.1

Originally the Stirling number \( s(10, 6) \) was given incorrectly as 6327. The correct number is 63273.

<table>
<thead>
<tr>
<th>( n )</th>
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<td>63273</td>
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<td>1</td>
</tr>
</tbody>
</table>

Reported 2013-11-25 by Svante Janson.

Equation (31.8.5)

31.8.5 \( \Psi_{1,-1} = (z^2 + (\lambda + 3a + 3)z + a) / z^3 \)

Originally the first term on the right side of the equation for \( \Psi_{1,-1} \) was \( z^3 \). The correct factor is \( z^2 \).

Reported 2013-07-25 by Christopher Künstler.

Equation (31.12.3)

31.12.3 \( \frac{d^2w}{dz^2} - \left( \frac{\gamma}{z} + \delta + z \right) \frac{dw}{dz} + \frac{\alpha z - q}{z} w = 0 \)

Originally the sign in front of the second term in this equation was +. The correct sign is −.

Reported 2013-10-31 by Henryk Witek.

Equation (34.4.2)

\[ \begin{vmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{vmatrix} = \Delta(j_1j_2j_3)\Delta(j_1l_2l_3)\Delta(l_1j_2l_3)\Delta(l_1l_2j_3) \]

34.4.2

\[ \times \sum_s \frac{(-1)^s(s+1)!}{(s-j_1-j_2-j_3)!(s-j_1-l_2-l_3)!(s-l_1-j_2-l_3)!(s-l_1-l_2-j_3)!} \]

\[ \times \frac{1}{(j_1+j_2+l_1+l_2-s)!(j_2+j_3+l_2+l_3-s)!(j_3+j_4+l_3+l_1-s)!} \]

Originally the factor \( \Delta(j_1j_2j_3)\Delta(j_1l_2l_3)\Delta(l_1j_2l_3)\Delta(l_1l_2j_3) \) was missing in this equation.

Reported 2012-12-31 by Yu Lin.
Other Changes

- Equations (4.45.8) and (4.45.9) have been replaced with equations that are better for numerically computing $\arctan x$.
- A new Subsection 13.29(v) Continued Fractions, has been added to cover computation of confluent hypergeometric functions by continued fractions.
- A new Subsection 14.5(vi) Addendum to §14.5(ii) $\mu = 0, \nu = 2$, containing the values of Legendre and Ferrers functions for degree $\nu = 2$ has been added.
- Subsection 14.18(iii) has been altered to identify Equations (14.18.6) and (14.18.7) as Christoffel’s Formulas.
- A new Subsection 15.19(v) Continued Fractions, has been added to cover computation of the Gauss hypergeometric functions by continued fractions.
- Special cases of normalization of Jacobi polynomials for which the general formula is undefined have been stated explicitly in Table 18.3.1.
- Bibliographic citations have been added in §§4.13, 5.6(i), 5.11(iii), 7.25(iii), 8.13(i), 10.37, 12.18, 14.11, 15.12(ii), 16.6, 18.16(ii), 18.16(iv), 18.24, 18.27(iv), 18.27(v), 18.28(i), 24.13(i), 28.36(iii).
- Cross-references have been added in §§1.2(i), 10.19(iii), 10.23(ii), 17.2(iii), 18.15(iii), 19.2(iv), 19.16(i).
- Several small revisions have been made. For details see §§5.11(ii), 10.12, 10.19(ii), 18.9(i), 18.16(iv), 19.7(ii), 22.2, 32.11(v), 32.13(ii).
- Entries for the Sage computational system have been updated in the online Software Cross Index.
- The default document format for DLMF is now HTML5 which includes MathML providing better accessibility and display of mathematics.
- All interactive 3D graphics on the DLMF website have been recast using WebGL and X3DOM, improving portability and performance; WebGL it is now the default format.

Version 1.0.5 (October 1, 2012)

Subsection 1.2(i)

The condition for (1.2.2), (1.2.4), and (1.2.5) was corrected. These equations are true only if $n$ is a positive integer. Previously $n$ was allowed to be zero.

Reported 2011-08-10 by Michael Somos.

Subsection 8.17(i)

The condition for the validity of (8.17.5) is that $m$ and $n$ are positive integers and $0 \leq x < 1$. Previously, no conditions were stated.

Reported 2011-03-23 by Stephen Bourn.

Equation (10.20.14)

\begin{align*}
10.20.14 \quad B_3(0) &= -\frac{959 717 11 846 03}{25 476 66 371 25 000 000} \times 2^{\frac{3}{2}} \\
\text{Originally this coefficient was given incorrectly as } B_3(0) &= -\frac{430 990 56 393 68 592 53}{5 681 67 34 399 42 500 000 000} \times 2^{\frac{3}{2}}. \text{ The other coefficients in this equation have not been changed.} \\
\text{Reported 2012-05-11 by Antony Lee.}
\end{align*}

Equation (13.16.4)

The condition for the validity of this equation is $\Re(\kappa - \mu) - \frac{1}{2} < 0$. Originally it was given incorrectly as $\Re(\kappa - \mu) - \frac{1}{2} > 0$.

Subsection 14.2(ii)

Originally it was stated, incorrectly, that $Q_\nu(x)$ is real when $\nu, \mu \in \mathbb{R}$ and $x \in (1, \infty)$. This statement is true only for $P_\nu(x)$ and $Q_\nu(x)$.

Reported 2012-07-18 by Hans Volkmer and Howard Cohl.

Equation (21.3.4)

\begin{align*}
21.3.4 \quad \theta \left[ \frac{\alpha + m_1}{\beta + m_2} \right] (z|\Omega) &= e^{2 \pi i \alpha \cdot m_2} \theta \left[ \frac{\alpha}{\beta} \right] (z|\Omega) \\
\text{Originally the vector } m_2 \text{ on the right-hand side was given incorrectly as } m_1. \\
\text{Reported 2012-08-27 by Klaas Vantournhout.}
\end{align*}
Subsection 21.10(i)

The entire original content of this subsection has been replaced by a reference.

Figures 22.3.22 and 22.3.23

The captions for these figures have been corrected to read, in part, “as a function of $k^2 = i\kappa^2$” (instead of $k^2 = i\kappa$). Also, the resolution of the graph in Figure 22.3.22 was improved near $\kappa = 3$.

Reported 2011-10-30 by Paul Abbott.

Equation (23.2.4)

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \mathbb{L} \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

Originally the denominator $(z-w)^2$ was given incorrectly as $(z-w^2)$.

Reported 2012-02-16 by James D. Walker.

Equation (24.4.26)

This equation is true only for $n > 0$. Previously, $n = 0$ was also allowed.

Reported 2012-05-14 by Vladimir Yurovsky.

Equation (26.12.26)

$$pp(n) \sim \frac{\left(\zeta(3)\right)^{7/36}}{2^{11/36} (3\pi)^{1/2} n^{25/36}} \times \exp \left( 3 \left( \frac{\zeta(3)}{\frac{1}{2} n} \right)^{1/3} + \zeta'(-1) \right)$$

Originally this equation was given incorrectly as

$$pp(n) \sim \left( \frac{\zeta(3)}{2^{11/25} n^{25}} \right)^{1/36} \times \exp \left( 3 \left( \frac{\zeta(3)n^2}{4} \right)^{1/3} + \zeta'(-1) \right).$$

Reported 2011-09-05 by Suresh Govindarajan.

Other Changes

- On August 24, 2012 Dr. Adri B. Olde Daalhuis was added as Mathematics Editor. This addition has been recorded at the end of the Preface (p. ix et seq.)
- Bibliographic citations were added in §§5.5(iii), 5.6(i), 5.10, 5.21, 7.13(ii), 10.19(iii), 10.21(i), 10.21(iv), 10.21(xiii), 10.21(xiv), 10.42, 10.46, 10.74(vii), 13.8(ii), 13.9(i), 13.9(ii), 13.11, 13.29(iv), 14.11, 15.13, 15.19(i), 17.18, 18.16(ii), 18.16(iv), 18.26(v), 19.12, 19.36(iv), 20.7(i), 20.7(ii), 20.7(iii), 20.7(vii), 25.11(iv), 25.18(i), 26.12(iv), 28.24, 28.34(ii), 29.20(i), 31.17(ii), 32.17, and as a general reference in Chapter 3.
- A cross-reference was added in §21.2(i).
- Several new equations have been added. See (8.17.24), (20.7.34), §20.11(v), (26.12.27), (36.2.28), and (36.2.29).
- The upper and lower bounds given in Equations (18.16.12) and (18.16.13) have been replaced with stronger bounds.
- Textual clarifications were made in §§5.5(ii), 7.13(ii), 15.6, 19.12, 20.7(iv), 21.2(i), 30.13(i), 30.14(i), and 31.17(ii).
- Other minor changes were made in the bibliography and index.

Version 1.0.4 (March 23, 2012)

Several minor improvements were made affecting display of math and graphics on the website; the software index and help files were updated.

Version 1.0.3 (Aug 29, 2011)

Equation (13.18.7)

$$W_{-\frac{1}{n},-\frac{1}{4}}(z^2) = e^{\frac{1}{2} z^2} \sqrt{\pi z} \text{erfc}(z)$$

Originally the left-hand side was given correctly as $W_{-\frac{1}{n},-\frac{1}{4}}(z^2)$; the equation is true also for $W_{-\frac{1}{n},+\frac{1}{4}}(z^2)$.

Other Changes

Bibliographic citations were added in §§3.5(iv), 4.44, 8.22(ii), 22.4(i), and minor clarifications were made in §§19.12, 20.7(vii), 22.9(i). In addition, several minor improvements were made affecting only ancilliary documents and links in the online version.

Version 1.0.2 (July 1, 2011)

Several minor improvements were made affecting display on the website; the help files were revised.
Version 1.0.1 (June 27, 2011)

Subsections 1.15(vi) and 1.15(vii)

The formulas in these subsections are valid only for \( x \geq 0 \). No conditions on \( x \) were given originally.

Reported 2010-10-18 by Andreas Kurt Richter.

Figure 10.48.5

Originally the ordinate labels 2 and 4 in this figure were placed too high.

Reported 2010-11-08 by Wolfgang Ehrhardt.

Equation (14.19.2)

\[
P_{\nu-\frac{1}{2}}^{\mu}(\cosh \xi) = \frac{\Gamma\left(\frac{1}{2} - \mu\right)}{\pi^{1/2} (1 - e^{-2\xi})^{\mu} e^{(\nu+(1/2))\xi}} \times \mathbf{F}\left(\frac{1}{2} - \mu, \frac{1}{2} + \nu - \mu, 1 - 2\mu, 1 - e^{-2\xi}\right),
\]

\[
\mu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots
\]

Originally the argument to \( \mathbf{F} \) in this equation was incorrect \( (e^{-2\xi}, \text{rather than } 1 - e^{-2\xi}) \), and the condition on \( \mu \) was too weak \( (\mu \neq \frac{1}{2}, \text{rather than } \mu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots) \). Also, the factor multiplying \( \mathbf{F} \) was rewritten to clarify the poles; originally it was 

\[
\frac{\Gamma(1-2\mu)2^{2\mu}}{\Gamma(1-\mu)(1-e^{-2\xi})^{\mu} e^{(\nu+(1/2))\xi}}.
\]

Reported 2010-11-02 by Alvaro Valenzuela.

Equation (17.13.3)

\[
\int_0^\infty t^{\alpha-1} \left(-t^{\alpha+\beta}; q\right)_\infty \, dt
\]

\[
17.13.3
\]

\[
= \frac{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma_q(\beta)}{\Gamma_q(1-\alpha)\Gamma_q(\alpha+\beta)}
\]

Originally the differential was identified incorrectly as \( d_q t \); the correct differential is \( d t \).

Reported 2011-04-08.

Table 18.9.1

The coefficient \( A_n \) for \( C_n^{(\lambda)}(x) \) in the first row of this table originally omitted the parentheses and was given as \( \frac{2^{n+\lambda}}{n+\frac{1}{2}} \), instead of \( \frac{2^{n+\lambda}}{n+\frac{1}{2}} \).

<table>
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<th>( p_n(x) )</th>
<th>( A_n )</th>
<th>( B_n )</th>
<th>( C_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_n^{(\lambda)}(x) )</td>
<td>( \frac{2^{n+\lambda}}{n+\frac{1}{2}} )</td>
<td>0</td>
<td>( \frac{n+2\lambda-1}{n+\frac{1}{2}} )</td>
</tr>
<tr>
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<tr>
<td>( U_n(x) )</td>
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<td>0</td>
<td>1</td>
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<tr>
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<td>( 2\delta_{n,0} )</td>
<td>( -2 + \delta_{n,0} )</td>
</tr>
<tr>
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<td>1</td>
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<td>( P_n(x) )</td>
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<td>( \frac{n}{n+\frac{1}{2}} )</td>
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</tr>
<tr>
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<td>( H_n^*(x) )</td>
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<td>0</td>
<td>( n )</td>
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Reported 2010-09-16 by Kendall Atkinson.

Subsection 19.16(iii)

Originally it was implied that \( R_C(x, y) \) is an elliptic integral. It was clarified that \( R_{-q}(b; z) \) is an elliptic integral iff the stated conditions hold; originally these conditions were stated as sufficient but not necessary. In particular, \( R_C(x, y) \) does not satisfy these conditions.

Reported 2010-11-23.

Table 22.5.4

Originally the limiting form for \( \text{sc}(z, k) \) in the last line of this table was incorrect \( (\cosh z, \text{instead of sinh} z) \).
Errata

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<th>sn(z, k) → tanh z</th>
<th>cd(z, k) → 1</th>
<th>dc(z, k) → 1</th>
<th>ns(z, k) → coth z</th>
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</thead>
<tbody>
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<td>cn(z, k) → sech z</td>
<td>sd(z, k) → sinh z</td>
<td>nc(z, k) → cosh z</td>
<td>ds(z, k) → csch z</td>
</tr>
<tr>
<td>dn(z, k) → sech z</td>
<td>nd(z, k) → cosh z</td>
<td>sc(z, k) → sinh z</td>
<td>cs(z, k) → csch z</td>
</tr>
</tbody>
</table>

Reported 2010-11-23.

Equation (22.16.14)

\[ E(x, k) = \int_{0}^{\sn(x, k)} \sqrt{\frac{1 - k^{2}t^{2}}{1 - t^{2}}} \, dt \]

Originally this equation appeared with the upper limit of integration as \( x \), rather than \( \sn(x, k) \).

Reported 2010-07-08 by Charles Karney.

Equation (26.7.6)

\[ B(n + 1) = \sum_{k=0}^{n} \binom{n}{k} B(k) \]

Originally this equation appeared with \( B(n) \) in the summation, instead of \( B(k) \).

Reported 2010-11-07 by Layne Watson.

Equation (36.10.14)

\[ 3 \left( \frac{\partial^{2} \Psi^{(E)}}{\partial x^{2}} - \frac{\partial^{2} \Psi^{(E)}}{\partial y^{2}} \right) + 2iz \frac{\partial \Psi^{(E)}}{\partial x} - x \Psi^{(E)} = 0 \]

Originally this equation appeared with \( \frac{\partial \Psi^{(n)}}{\partial x} \) in the second term, rather than \( \frac{\partial \Psi^{(E)}}{\partial x} \).

Reported 2010-04-02.

Other Changes

- The general references for each chapter were inserted under the \( i \)-symbol on the chapter title pages. Originally these appeared only in the References sections of the individual chapters in the Handbook.
- The definition of \( R_{C}(x, y) \) was revised in Notations beginning on p. 909.
- Additions and revisions were made in the Cross Index for Computing Special Functions at http://dlmf.nist.gov/software/.

To see the effect of these changes, see http://dlmf.nist.gov/.

Version 1.0.0 (May 7, 2010)

The Handbook of Mathematical Functions was published, and the Digital Library of Mathematical Functions was released.

Bibliography


