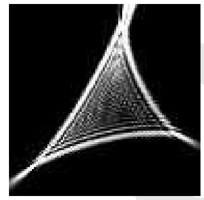
Chapter AI:

Airy and Related Functions

Frank W. J. Olver[1]



[1]Institute for Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, MD 20742. olver@bessel.umd.edu

Acknowledgments: This chapter is based in part on Abramowitz and Stegun (1964, Chapter 10) by H. A. Antosiewicz. The graphs and visualizations in §AI.3 were constructed by R. R. Lipman, D. W. Lozier, B. V. Saunders, Q. Wang with the aid of the software package of Amos (1986). In §AI.9, B. R. Fabijonas supplied higher terms in some of the asymptotic expansions and computed the numerical tables. D. W. Lozier assisted with the compilation of §AI.21. The Web site was prepared by R. F. Boisvert and B. R. Miller.

Status: Draft; Format: Print. Draft of 2003-09-05 17:04; Printed 2003-09-10 20:08.
Version info: DocClass: 2002/04/15 DLMF Chapter; AI.tex: 1.26; figs/rainbow.eps: 1.1; figs/cusp.eps: 1.1; figs/aisurf.eps: 1.1; figs/bisurf.eps: 1.1; figs/rainbow_thumb.eps: 1.1; figs/cusp_thumb.eps: 1.1; 2D/aibim_math.eps: 1.2; 2D/aipbipn_math.eps: 1.2; 2D/ai131_math.eps: 1.1; 2D/ai132_math.eps: 1.1; 3D/Ai.ps: 1.2; 3D/Bi.ps: 1.2; 3D/reAi.ps: 1.2; 3D/reBi.ps: 1.2; 3D/imAi.ps: 1.2; 3D/imBi.ps: 1.2; 3D/Aip.ps: 1.2; 3D/Bip.ps: 1.2; 3D/reAip.ps: 1.2; 3D/reBip.ps: 1.2; 3D/imAip.ps: 1.2; 3D/imBip.ps: 1.2.

Notation

AI.1 Notation

x real variable,
z complex variable,
$\mathbb{C}$ complex plane (excluding infinity),
$\mathbb{R}$ real line (excluding infinity).

Unless otherwise noted, primes indicate a derivative with respect to the argument.

Airy Functions The notation

AI.1.1 $$\mathrm{Ai}(x),\quad \mathrm{Bi}(x),$$

is due to Jeffreys (1928). Later, the sign of x was changed.

Another notation is that of Fock (1945):

AI.1.2 $$U(x) = \sqrt{\pi}\,\mathrm{Bi}(x),\quad V(x) = \sqrt{\pi}\,\mathrm{Ai}(x).$$

Scorer Functions (See §AI.12) The notation

AI.1.3 $$\mathrm{Gi}(x),\quad \mathrm{Hi}(x)$$

is due to Scorer (1950). Another notation is that of Tumarkin (1959):

AI.1.4 $$e_0(x) = \pi\,\mathrm{Hi}(-x),\quad \tilde{e}_0(x) = -\pi\,\mathrm{Gi}(-x).$$

Mathematical Properties

AI.2 Differential equation

AI.2(i) Airy's Equation

AI.2.1 $$\frac{d^2w}{dz^2} = zw.$$

Standard solutions are

AI.2.2 $$w = \mathrm{Ai}(z),\quad \mathrm{Bi}(z),\quad \mathrm{Ai}(ze^{-2\pi i/3}),\quad \mathrm{Ai}(ze^{2\pi i/3}).$$

All solutions are entire functions of z.

AI.2(ii) Initial Values

AI.2.3 $$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} = 0.35502\,80539,$$

AI.2.4 $$\mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(\frac{1}{3})} = -0.25881\,94038,$$

AI.2.5 $$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma(\frac{2}{3})} = 0.61492\,66276,$$

AI.2.6 $$\mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(\frac{1}{3})} = 0.44828\,83574.$$

AI.2(iii) Numerically Satisfactory Pairs of Solutions

Pairs		*Region*
$\mathrm{Ai}(x)$	$\mathrm{Bi}(x)$	$x \in \mathbb{R}$
$\mathrm{Ai}(z)$	$\mathrm{Bi}(z)$	$\lvert \mathrm{ph}\, z\rvert \le \frac{1}{3}\pi$ (and along $x < 0$)
$\mathrm{Ai}(z)$	$\mathrm{Ai}(ze^{-2\pi i/3})$	$-\frac{1}{3}\pi \le \mathrm{ph}\, z \le \pi.$
$\mathrm{Ai}(z)$	$\mathrm{Ai}(ze^{2\pi i/3})$	$-\pi \le \mathrm{ph}\, z \le \frac{1}{3}\pi.$
$\mathrm{Ai}(ze^{-2\pi i/3})$	$\mathrm{Ai}(ze^{2\pi i/3})$	$\lvert \mathrm{ph}(-z)\rvert \le \frac{2}{3}\pi.$

For graphical interpretations, see Figures AI.3.1–AI.3.2 and Figures AI.3.3–AI.3.4, for the real and complex cases, respectively. ⇒ DLMF

AI.2(iv) Wronskians

AI.2.7 $$\mathscr{W}\left\{\mathrm{Ai}(z), \mathrm{Bi}(z)\right\} = \frac{1}{\pi},$$

AI.2.8 $$\mathscr{W}\left\{\mathrm{Ai}(z), \mathrm{Ai}(ze^{\mp 2\pi i/3})\right\} = \frac{e^{\pm\pi i/6}}{2\pi},$$

AI.2.9 $$\mathscr{W}\left\{\mathrm{Ai}(ze^{-2\pi i/3}), \mathrm{Ai}(ze^{2\pi i/3})\right\} = \frac{1}{2\pi i}.$$

AI.2(v) Connection Formulas

AI.2.10 $$\mathrm{Bi}(z) = e^{-\pi i/6}\,\mathrm{Ai}(ze^{-2\pi i/3}) + e^{\pi i/6}\,\mathrm{Ai}(ze^{2\pi i/3}),$$

AI.2.11 $$\mathrm{Ai}(ze^{\mp 2\pi i/3}) = \tfrac{1}{2}e^{\mp\pi i/3}\left(\mathrm{Ai}(z) \pm i\,\mathrm{Bi}(z)\right),$$

AI.2.12 $$\mathrm{Ai}(z) + e^{-2\pi i/3}\,\mathrm{Ai}(ze^{-2\pi i/3}) + e^{2\pi i/3}\,\mathrm{Ai}(ze^{2\pi i/3}) = 0,$$

AI.2.13 $$\mathrm{Bi}(z) + e^{-2\pi i/3}\,\mathrm{Bi}(ze^{-2\pi i/3}) + e^{2\pi i/3}\,\mathrm{Bi}(ze^{2\pi i/3}) = 0.$$

AI.2(vi) Riccati Form of Differential Equation

AI.2.14 $$\frac{dW}{dz} + W^2 = z.$$

This is satisfied by $W = (1/w)dw/dz$, where w is any solution of (AI.2.1).

AI.3 Graphs and Visualizations

AI.3(i) Real Variable

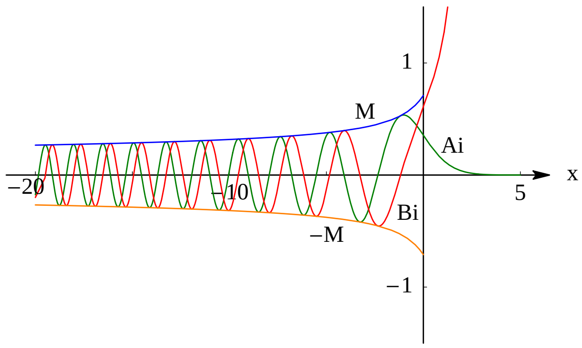


Figure AI.3.1: $\mathrm{Ai}(x)$, $\mathrm{Bi}(x)$, $M(x)$.

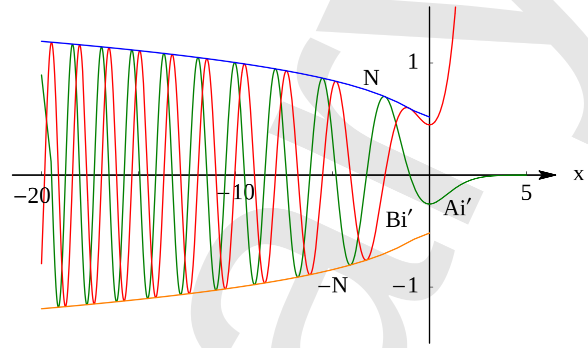
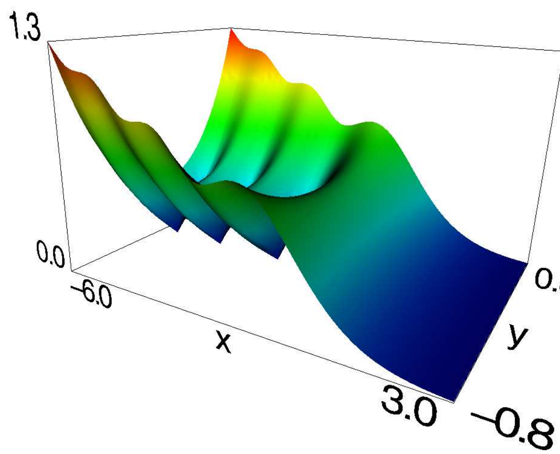


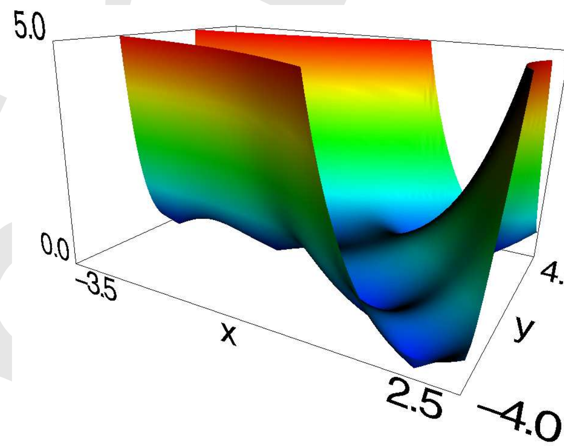
Figure AI.3.2: $\mathrm{Ai}'(x)$, $\mathrm{Bi}'(x)$, $N(x)$.

AI.3(ii) Complex Variable



DLMF ⇐

Figure AI.3.3: $|\,\mathrm{Ai}(x+iy)|$.



DLMF ⇐

Figure AI.3.4: $|\,\mathrm{Bi}(x+iy)|$.

AI.4 Maclaurin Series

For $z \in \mathbb{C}$

AI.4.1

$$\mathrm{Ai}(z) = \mathrm{Ai}(0)\left(1 + \tfrac{1}{3!}z^3 + \tfrac{1\cdot 4}{6!}z^6 + \tfrac{1\cdot 4\cdot 7}{9!}z^9 + \cdots\right) + \mathrm{Ai}'(0)\left(z + \tfrac{2}{4!}z^4 + \tfrac{2\cdot 5}{7!}z^7 + \tfrac{2\cdot 5\cdot 8}{10!}z^{10} + \cdots\right),$$

AI.4.2

$$\mathrm{Ai}'(z) = \mathrm{Ai}'(0)\left(1 + \tfrac{2}{3!}z^3 + \tfrac{2\cdot 5}{6!}z^6 + \tfrac{2\cdot 5\cdot 8}{9!}z^9 + \cdots\right) + \mathrm{Ai}(0)\left(\tfrac{1}{2!}z^2 + \tfrac{1\cdot 4}{5!}z^5 + \tfrac{1\cdot 4\cdot 7}{8!}z^8 + \cdots\right),$$

AI.4.3

$$\mathrm{Bi}(z) = \mathrm{Bi}(0)(1 + \tfrac{1}{3!}z^3 + \tfrac{1\cdot 4}{6!}z^6 + \tfrac{1\cdot 4\cdot 7}{9!}z^9 + \cdots) + \mathrm{Bi}'(0)(z + \tfrac{2}{4!}z^4 + \tfrac{2\cdot 5}{7!}z^7 + \tfrac{2\cdot 5\cdot 8}{10!}z^{10} + \cdots),$$

AI.4.4

$$\mathrm{Bi}'(z) = \mathrm{Bi}'(0)(1 + \tfrac{2}{3!}z^3 + \tfrac{2\cdot 5}{6!}z^6 + \tfrac{2\cdot 5\cdot 8}{9!}z^9 + \cdots) + \mathrm{Bi}(0)(\tfrac{1}{2!}z^2 + \tfrac{1\cdot 4}{5!}z^5 + \tfrac{1\cdot 4\cdot 7}{8!}z^8 + \cdots).$$

AI.5 Integral Representations

AI.5(i) Real Variable

AI.5.1 $$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}t^3 + xt)\,dt,$$

AI.5.2 $$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \exp(-\tfrac{1}{3}t^3 + xt)\,dt + \frac{1}{\pi}\int_0^\infty \sin(\tfrac{1}{3}t^3 + xt)\,dt.$$

AI.5(ii) Complex Variable

AI.5.3 $$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp(\tfrac{1}{3}t^3 - zt)\,dt,$$

AI.5.4 $$\mathrm{Ai}'(z) = -\frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} t\exp(\tfrac{1}{3}t^3 - zt)\,dt,$$

AI.5.5
$$\mathrm{Bi}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty e^{\pi i/3}} \exp(\tfrac{1}{3}t^3 - zt)\,dt + \frac{1}{2\pi}\int_{-\infty}^{\infty e^{-\pi i/3}} \exp(\tfrac{1}{3}t^3 - zt)\,dt,$$

AI.5.6
$$\mathrm{Bi}'(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty e^{\pi i/3}} t\exp(\tfrac{1}{3}t^3 - zt)\,dt - \frac{1}{2\pi}\int_{-\infty}^{\infty e^{-\pi i/3}} t\exp(\tfrac{1}{3}t^3 - zt)\,dt.$$

See also (AI.10.17) and (AI.10.18).

AI.6 Relations to Other Functions

AI.6(i) Airy Functions as Bessel Functions

AI.6.1
$$\zeta = \tfrac{2}{3}z^{3/2}.$$

AI.6.2
$$\begin{aligned}
\mathrm{Ai}(z) &= \pi^{-1}\sqrt{z/3}\,K_{\pm 1/3}(\zeta)\\
&= \tfrac{1}{3}\sqrt{z}\left(I_{-1/3}(\zeta) - I_{1/3}(\zeta)\right)\\
&= \tfrac{1}{2}\sqrt{z/3}e^{2\pi i/3}\,H^{(1)}_{1/3}(\zeta e^{\pi i/2})\\
&= \tfrac{1}{2}\sqrt{z/3}e^{\pi i/3}\,H^{(1)}_{-1/3}(\zeta e^{\pi i/2})\\
&= \tfrac{1}{2}\sqrt{z/3}e^{-2\pi i/3}\,H^{(2)}_{1/3}(\zeta e^{-\pi i/2})\\
&= \tfrac{1}{2}\sqrt{z/3}e^{-\pi i/3}\,H^{(2)}_{-1/3}(\zeta e^{-\pi i/2}),
\end{aligned}$$

AI.6.3
$$\begin{aligned}
\mathrm{Ai}'(z) &= -\pi^{-1}(z/\sqrt{3})\,K_{\pm 2/3}(\zeta)\\
&= (z/3)\left(I_{2/3}(\zeta) - I_{-2/3}(\zeta)\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})e^{-\pi i/6}\,H^{(1)}_{2/3}(\zeta e^{\pi i/2})\\
&= \tfrac{1}{2}(z/\sqrt{3})e^{-5\pi i/6}\,H^{(1)}_{-2/3}(\zeta e^{\pi i/2})\\
&= \tfrac{1}{2}(z/\sqrt{3})e^{\pi i/6}\,H^{(2)}_{2/3}(\zeta e^{-\pi i/2})\\
&= \tfrac{1}{2}(z/\sqrt{3})e^{5\pi i/6}\,H^{(2)}_{-2/3}(\zeta e^{-\pi i/2}),
\end{aligned}$$

AI.6.4
$$\begin{aligned}
\mathrm{Bi}(z) &= \sqrt{z/3}\left(I_{1/3}(\zeta) + I_{-1/3}(\zeta)\right)\\
&= \tfrac{1}{2}\sqrt{z/3}\left(e^{\pi i/6}\,H^{(1)}_{1/3}(\zeta e^{-\pi i/2}) + e^{-\pi i/6}\,H^{(2)}_{1/3}(\zeta e^{\pi i/2})\right)\\
&= \tfrac{1}{2}\sqrt{z/3}\left(e^{-\pi i/6}\,H^{(1)}_{-1/3}(\zeta e^{-\pi i/2}) + e^{\pi i/6}\,H^{(2)}_{-1/3}(\zeta e^{\pi i/2})\right),
\end{aligned}$$

AI.6.5
$$\begin{aligned}
\mathrm{Bi}'(z) &= (z/\sqrt{3})\left(I_{2/3}(\zeta) + I_{-2/3}(\zeta)\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})\left(e^{\pi i/3}\,H^{(1)}_{2/3}(\zeta e^{-\pi i/2}) + e^{-\pi i/3}\,H^{(2)}_{2/3}(\zeta e^{\pi i/2})\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})\left(e^{-\pi i/3}\,H^{(1)}_{-2/3}(\zeta e^{-\pi i/2}) + e^{\pi i/3}\,H^{(2)}_{-2/3}(\zeta e^{\pi i/2})\right),
\end{aligned}$$

AI.6.6
$$\begin{aligned}
\mathrm{Ai}(-z) &= (\sqrt{z}/3)\left(J_{1/3}(\zeta) + J_{-1/3}(\zeta)\right)\\
&= \tfrac{1}{2}\sqrt{z/3}\left(e^{\pi i/6}\,H^{(1)}_{1/3}(\zeta) + e^{-\pi i/6}\,H^{(2)}_{1/3}(\zeta)\right)\\
&= \tfrac{1}{2}\sqrt{z/3}\left(e^{-\pi i/6}\,H^{(1)}_{-1/3}(\zeta) + e^{\pi i/6}\,H^{(2)}_{-1/3}(\zeta)\right),
\end{aligned}$$

AI.6.7
$$\begin{aligned}
\mathrm{Ai}'(-z) &= (z/3)\left(J_{2/3}(\zeta) - J_{-2/3}(\zeta)\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})\left(e^{-\pi i/6}\,H^{(1)}_{2/3}(\zeta) + e^{\pi i/6}\,H^{(2)}_{2/3}(\zeta)\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})\left(e^{-5\pi i/6}\,H^{(1)}_{-2/3}(\zeta) + e^{5\pi i/6}\,H^{(2)}_{-2/3}(\zeta)\right),
\end{aligned}$$

AI.6.8
$$\begin{aligned}
\mathrm{Bi}(-z) &= \sqrt{z/3}\left(J_{-1/3}(\zeta) - J_{1/3}(\zeta)\right)\\
&= \tfrac{1}{2}\sqrt{z/3}\left(e^{2\pi i/3}\,H^{(1)}_{1/3}(\zeta) + e^{-2\pi i/3}\,H^{(2)}_{1/3}(\zeta)\right)\\
&= \tfrac{1}{2}\sqrt{z/3}\left(e^{\pi i/3}\,H^{(1)}_{-1/3}(\zeta) + e^{-\pi i/3}\,H^{(2)}_{-1/3}(\zeta)\right),
\end{aligned}$$

AI.6.9
$$\begin{aligned}
\mathrm{Bi}'(-z) &= (z/\sqrt{3})\left(J_{-2/3}(\zeta) + J_{2/3}(\zeta)\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})\left(e^{\pi i/3}\,H^{(1)}_{2/3}(\zeta) + e^{-\pi i/3}\,H^{(2)}_{2/3}(\zeta)\right)\\
&= \tfrac{1}{2}(z/\sqrt{3})\left(e^{-\pi i/3}\,H^{(1)}_{-2/3}(\zeta) + e^{\pi i/3}\,H^{(2)}_{-2/3}(\zeta)\right).
\end{aligned}$$

AI.6(ii) Bessel Functions as Airy Functions

AI.6.10
$$z = (\tfrac{3}{2}\zeta)^{2/3}.$$

AI.6.11
$$J_{\pm 1/3}(\zeta) = \tfrac{1}{2}\sqrt{3/z}\left(\sqrt{3}\,\mathrm{Ai}(-z) \mp \mathrm{Bi}(-z)\right),$$

AI.6.12
$$J_{\pm 2/3}(\zeta) = \tfrac{1}{2}(\sqrt{3}/z)\left(\pm\sqrt{3}\,\mathrm{Ai}'(-z) + \mathrm{Bi}'(-z)\right),$$

AI.6.13
$$I_{\pm 1/3}(\zeta) = \tfrac{1}{2}\sqrt{3/z}\left(\mp\sqrt{3}\,\mathrm{Ai}(z) + \mathrm{Bi}(z)\right),$$

AI.6.14
$$I_{\pm 2/3}(\zeta) = \tfrac{1}{2}(\sqrt{3}/z)\left(\pm\sqrt{3}\,\mathrm{Ai}'(z) + \mathrm{Bi}'(z)\right),$$

AI.6.15
$$K_{\pm 1/3}(\zeta) = \pi\sqrt{3/z}\,\mathrm{Ai}(z),$$

AI.6.16
$$K_{\pm 2/3}(\zeta) = -\pi(\sqrt{3}/z)\,\mathrm{Ai}'(z),$$

AI.6.17
$$\begin{aligned}
H^{(1)}_{1/3}(\zeta) &= e^{-\pi i/3}\,H^{(1)}_{-1/3}(\zeta)\\
&= e^{-\pi i/6}\sqrt{3/z}\left(\mathrm{Ai}(-z) - i\,\mathrm{Bi}(-z)\right),
\end{aligned}$$

AI.6.18
$$\begin{aligned}
H^{(1)}_{2/3}(\zeta) &= e^{-2\pi i/3}\,H^{(1)}_{-2/3}(\zeta)\\
&= e^{\pi i/6}(\sqrt{3}/z)\left(\mathrm{Ai}'(-z) - i\,\mathrm{Bi}'(-z)\right),
\end{aligned}$$

AI.6.19
$$H^{(2)}_{1/3}(\zeta) = e^{\pi i/3}\, H^{(2)}_{-1/3}(\zeta) = e^{\pi i/6}\sqrt{3/z}\,(\mathrm{Ai}(-z) + i\,\mathrm{Bi}(-z))\,,$$

AI.6.20
$$H^{(2)}_{2/3}(\zeta) = e^{2\pi i/3}\, H^{(2)}_{-2/3}(\zeta) = e^{-\pi i/6}(\sqrt{3}/z)\left(\mathrm{Ai}'(-z) + i\,\mathrm{Bi}'(-z)\right).$$

AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

AI.6.21
$$\zeta = \tfrac{2}{3} z^{3/2}.$$

AI.6.22
$$\mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4}\, W_{0,1/3}(2\zeta) = 3^{-1/6}\pi^{-1/2}\zeta^{2/3} e^{-\zeta}\, U(\tfrac{5}{6},\tfrac{5}{3},2\zeta),$$

AI.6.23
$$\mathrm{Ai}'(z) = -\tfrac{1}{2}\pi^{-1/2} z^{1/4}\, W_{0,2/3}(2\zeta) = -3^{1/6}\pi^{-1/2}\zeta^{4/3} e^{-\zeta}\, U(\tfrac{7}{6},\tfrac{7}{3},2\zeta),$$

AI.6.24
$$\mathrm{Bi}(z) = \frac{1}{2^{1/3}\Gamma(\frac{2}{3})} z^{-1/4}\, M_{0,-1/3}(2\zeta) + \frac{3}{2^{5/3}\Gamma(\frac{1}{3})} z^{-1/4}\, M_{0,1/3}(2\zeta),$$

AI.6.25
$$\mathrm{Bi}'(z) = \frac{2^{1/3}}{\Gamma(\frac{1}{3})} z^{1/4}\, M_{0,-2/3}(2\zeta) + \frac{3}{2^{10/3}\Gamma(\frac{2}{3})} z^{1/4}\, M_{0,2/3}(2\zeta),$$

AI.6.26
$$\mathrm{Bi}(z) = \frac{1}{3^{1/6}\Gamma(\frac{2}{3})} e^{-\zeta}\ {}_1F_1(\tfrac{1}{6};\tfrac{1}{3};2\zeta) + \frac{3^{5/6}}{2^{2/3}\Gamma(\frac{1}{3})}\zeta^{2/3} e^{-\zeta}\ {}_1F_1(\tfrac{5}{6};\tfrac{5}{3};2\zeta),$$

AI.6.27
$$\mathrm{Bi}'(z) = \frac{3^{1/6}}{\Gamma(\frac{1}{3})} e^{-\zeta}\ {}_1F_1(-\tfrac{1}{6};-\tfrac{1}{3};2\zeta) + \frac{3^{7/6}}{2^{7/3}\Gamma(\frac{2}{3})}\zeta^{4/3} e^{-\zeta}\ {}_1F_1(\tfrac{7}{6};\tfrac{7}{3};\zeta).$$

AI.7 Asymptotic Expansions

AI.7(i) Poincaré-Type Expansions

AI.7.1
$$\zeta = \tfrac{2}{3} z^{3/2}\,,\quad u_0 = 1\,,\quad v_0 = 1\,,$$
$$u_s = \frac{(2s+1)(2s+3)(2s+5)\cdots(6s-1)}{(216)^s s!}\,,$$
$$v_s = -\frac{6s+1}{6s-1} u_s\,.$$

AI.7.2
$$\mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi} z^{1/4}} \sum_{s=0}^{\infty} (-)^s \frac{u_s}{\zeta^s},\quad |\,\mathrm{ph}\, z| < \pi,$$

AI.7.3
$$\mathrm{Ai}'(z) \sim -\frac{z^{1/4} e^{-\zeta}}{2\sqrt{\pi}} \sum_{s=0}^{\infty} (-)^s \frac{v_s}{\zeta^s},\quad |\,\mathrm{ph}\, z| < \pi,$$

AI.7.4
$$\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} \left(\cos(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s}}{\zeta^{2s}} + \sin(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s+1}}{\zeta^{2s+1}}\right),\quad |\,\mathrm{ph}\, z| < \tfrac{2}{3}\pi,$$

AI.7.5
$$\mathrm{Ai}'(-z) \sim \frac{z^{1/4}}{\sqrt{\pi}} \left(\sin(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{v_{2s}}{\zeta^{2s}} - \cos(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{v_{2s+1}}{\zeta^{2s+1}}\right),\quad |\,\mathrm{ph}\, z| < \tfrac{2}{3}\pi,$$

AI.7.6
$$\mathrm{Bi}(z) \sim \frac{e^{\zeta}}{\sqrt{\pi} z^{1/4}} \sum_{s=0}^{\infty} \frac{u_s}{\zeta^s},\quad |\,\mathrm{ph}\, z| < \tfrac{1}{3}\pi,$$

AI.7.7
$$\mathrm{Bi}'(z) \sim \frac{z^{1/4} e^{\zeta}}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{v_s}{\zeta^s},\quad |\,\mathrm{ph}\, z| < \tfrac{1}{3}\pi,$$

AI.7.8
$$\mathrm{Bi}(-z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} \left(-\sin(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s}}{\zeta^{2s}} + \cos(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s+1}}{\zeta^{2s+1}}\right),\quad |\,\mathrm{ph}\, z| < \tfrac{2}{3}\pi,$$

AI.7.9
$$\mathrm{Bi}'(-z) \sim \frac{z^{1/4}}{\sqrt{\pi}} \left(\cos(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{v_{2s}}{\zeta^{2s}} + \sin(\zeta - \tfrac{\pi}{4}) \sum_{s=0}^{\infty} (-)^s \frac{v_{2s+1}}{\zeta^{2s+1}}\right),\quad |\,\mathrm{ph}\, z| < \tfrac{2}{3}\pi,$$

AI.7.10
$$\mathrm{Bi}(z e^{\pm\pi i/3}) \sim \sqrt{\frac{2}{\pi}} \frac{e^{\pm\pi i/6}}{z^{1/4}} \times \left(\cos(\zeta - \tfrac{\pi}{4} \mp \tfrac{i}{2}\ln 2) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s}}{\zeta^{2s}} + \sin(\zeta - \tfrac{\pi}{4} \mp \tfrac{i}{2}\ln 2) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s+1}}{\zeta^{2s+1}}\right),\quad |\,\mathrm{ph}\, z| < \tfrac{2}{3}\pi,$$

AI.7.11
$$\mathrm{Bi}'(z e^{\pm\pi i/3}) \sim \sqrt{\frac{2}{\pi}} e^{\mp\pi i/6} z^{1/4} \times \left(-\sin(\zeta - \tfrac{\pi}{4} \mp \tfrac{i}{2}\ln 2) \sum_{s=0}^{\infty} (-)^s \frac{v_{2s}}{\zeta^{2s}} + \cos(\zeta - \tfrac{\pi}{4} \mp \tfrac{i}{2}\ln 2) \sum_{s=0}^{\infty} (-)^s \frac{v_{2s+1}}{\zeta^{2s+1}}\right),\quad |\,\mathrm{ph}\, z| < \tfrac{2}{3}\pi.$$

AI.7(ii) Error Bounds for Real Arguments

In (AI.7.2) and (AI.7.3) the nth error term, that is, the error on truncating the expansion at n terms, is bounded in magnitude by the first neglected term and has the same sign, provided that the following term is of opposite sign, that is, if $n \geq 0$ for (AI.7.2) and $n \geq 1$ for (AI.7.3). As special cases

AI.7.12
$$\operatorname{Ai}(x) \leq \frac{e^{-\zeta}}{2\sqrt{\pi}x^{1/4}}, \qquad |\operatorname{Ai}'(x)| \leq \frac{x^{1/4}e^{-\zeta}}{2\sqrt{\pi}}\left(1+\frac{7}{72\zeta}\right),$$

when $0 \leq x \leq \infty$, where $\zeta = \frac{2}{3}x^{3/2}$.

In (AI.7.4), (AI.7.5), (AI.7.8), (AI.7.9) the nth error term in each infinite series is bounded in magnitude by the first neglected term and has the same sign, provided that the following term in the series is of opposite sign.

In (AI.7.6) and (AI.7.7) the nth error term is bounded in magnitude by the first neglected term multiplied by $2\chi(n)\exp(\sigma\pi/(72\zeta))$ where $\sigma = 5$ for (AI.7.6) and $\sigma = 7$ for (AI.7.7), provided that $n \geq 1$ in both cases. Here

AI.7.13
$$\chi(n) = \pi^{1/2}\Gamma(\tfrac{1}{2}n+1)/\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}),$$

(see table AI.7.1).

As special cases

AI.7.14
$$\operatorname{Bi}(x) \leq \frac{e^{\zeta}}{\sqrt{\pi}x^{1/4}}\left(1+\frac{5\pi}{72\zeta}\exp\left(\frac{5\pi}{72\zeta}\right)\right),$$

AI.7.15
$$\operatorname{Bi}'(x) \leq \frac{x^{1/4}e^{\zeta}}{\sqrt{\pi}}\left(1+\frac{7\pi}{72\zeta}\exp\left(\frac{7\pi}{72\zeta}\right)\right),$$

when $0 \leq x < \infty$, where $\zeta = \frac{2}{3}x^{3/2}$.

For large n, $\chi(n) \sim (\frac{1}{2}\pi n)^{1/2}$.

AI.7(iii) Error Bounds for Complex Arguments

In (AI.7.2) and (AI.7.3) the nth error term is bounded in magnitude by the first neglected term multiplied by

AI.7.16
$$2\exp\left(\frac{\sigma}{36|\zeta|}\right), \quad 2\chi(n)\exp\left(\frac{\sigma\pi}{72|\zeta|}\right) \quad \text{or} \quad \frac{4\chi(n)}{|\cos(\operatorname{ph}\zeta)|^n}\exp\left(\frac{\sigma\pi}{36|\Re\zeta|}\right),$$

according as $|\operatorname{ph} z| \leq \frac{1}{3}\pi$, $\frac{1}{3}\pi \leq |\operatorname{ph} z| \leq \frac{2}{3}\pi$, $\frac{2}{3}\pi \leq |\operatorname{ph} z| \leq \pi$. Here $\sigma = 5$ for (AI.7.2) and $\sigma = 7$ for (AI.7.3).

Corresponding bounds for the errors in (AI.7.4) to (AI.7.11) may be obtained by use of these results and connection formulas

AI.7.17
$$\operatorname{Ai}(-z) = e^{\pi i/3}\operatorname{Ai}(ze^{\pi i/3}) + e^{-\pi i/3}\operatorname{Ai}(ze^{-\pi i/3}),$$

AI.7.18
$$\operatorname{Ai}'(-z) = e^{-\pi i/3}\operatorname{Ai}'(ze^{\pi i/3}) + e^{\pi i/3}\operatorname{Ai}'(ze^{-\pi i/3}),$$

AI.7.19
$$\operatorname{Bi}(z) = e^{-\pi i/6}\operatorname{Ai}(ze^{-2\pi i/3}) + e^{\pi i/6}\operatorname{Ai}(ze^{2\pi i/3}),$$

AI.7.20
$$\operatorname{Bi}'(z) = e^{-5\pi i/6}\operatorname{Ai}'(ze^{-2\pi i/3}) + e^{5\pi i/6}\operatorname{Ai}'(ze^{2\pi i/3}),$$

with z in (AI.7.19) replaced by $-z$ in the case of (AI.7.8), (AI.7.9), and by $ze^{\pm\pi i/3}$ in the case of (AI.7.10), (AI.7.11).

AI.7(iv) Exponentially-Improved Expansions

In (AI.7.2) and (AI.7.3) let

AI.7.21
$$\operatorname{Ai}(z) = \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}}\left(\sum_{s=0}^{n-1}(-)^s\frac{u_s}{\zeta^s} + R_n(z)\right),$$

AI.7.22
$$\operatorname{Ai}'(z) = -\frac{z^{1/4}e^{-\zeta}}{2\sqrt{\pi}}\left(\sum_{s=0}^{n-1}(-)^s\frac{v_s}{\zeta^s} + S_n(z)\right),$$

with $n = \operatorname{int}[2|\zeta|]$. Then

AI.7.23
$$R_n(z) = (-)^n\sum_{s=0}^{m-1}(-)^s u_s\frac{G_{n-s}(2\zeta)}{\zeta^s} + R_{m,n}(z),$$

AI.7.24
$$S_n(z) = (-)^{n-1}\sum_{s=0}^{m-1}(-)^s v_s\frac{G_{n-s}(2\zeta)}{\zeta^s} + S_{m,n}(z),$$

where

AI.7.25
$$G_p(z) = \frac{e^z}{2\pi}\Gamma(p)\Gamma(1-p,z).$$

(For the incomplete Gamma function see Chapter IG.) And as $z \to \infty$ with m fixed

AI.7.26
$$R_{m,n}(z), \quad S_{m,n}(z) = O(e^{-2|\zeta|}\zeta^{-m}), \quad |\operatorname{ph} z| \leq \tfrac{2}{3}\pi.$$

For re-expansions of the remainder terms in (AI.7.4)–(AI.7.11) use the above results and (AI.7.17), (AI.7.19).

For higher re-expansions of the remainder terms see Olde Daalhuis (1995), Olde Daalhuis (1996), and Olde Daalhuis and Olver (1995).

AI.8 Modulus and Phase

AI.8(i) Definitions

In this section x is real and nonpositive.

AI.8.1
$$\operatorname{Ai}(x) = M(x)\sin\theta(x),$$

AI.8.2
$$\operatorname{Bi}(x) = M(x)\cos\theta(x),$$

AI.8.3
$$M(x) = \sqrt{\operatorname{Ai}^2(x) + \operatorname{Bi}^2(x)},$$

AI.8.4
$$\theta(x) = \arctan\left(\operatorname{Ai}(x)/\operatorname{Bi}(x)\right),$$

AI.8.5
$$\operatorname{Ai}'(x) = N(x)\sin\phi(x),$$

Table AI.7.1: Error bound factors for real arguments.

n	1	2	3	4	5	6	7	8	9	10
$\chi(n)$	1.57	2.00	2.36	2.67	2.95	3.20	3.44	3.66	3.87	4.06
n	11	12	13	14	15	16	17	18	19	20
$\chi(n)$	4.25	4.43	4.61	4.77	4.94	5.09	5.24	5.39	5.54	5.68

AI.8.6 $$\mathrm{Bi}'(x) = N(x)\cos\phi(x),$$

AI.8.7 $$N(x) = \sqrt{\mathrm{Ai}'^2(x) + \mathrm{Bi}'^2(x)},$$

AI.8.8 $$\phi(x) = \arctan\left(\mathrm{Ai}'(x)/\,\mathrm{Bi}'(x)\right).$$

See §AI.3(i) for graphs.

The branches of $\theta(x)$ and $\phi(x)$ are continuous and fixed by $\theta(0) = -\phi(0) = \frac{1}{6}\pi$. (These definitions of $\theta(x)$ and $\phi(x)$ differ from Abramowitz and Stegun (1964, Chapter 10), and agree more closely with those used in Miller (1946) and Olver (1974, Chapter 11).)

AI.8(ii) Identities

Primes denote differentiations with respect to x, which is again assumed to be real and nonpositive.

AI.8.9 $$M\,N\sin(\theta-\phi) = \pi^{-1} \text{ (Wronskian)},$$

AI.8.10 $$M^2\,\theta' = -\pi^{-1}, \quad N^2\,\phi' = \pi^{-1}x, \quad N\,N' = x\,M\,M',$$

AI.8.11 $$N^2 = M'^2 + M^2\,\theta'^2 = M'^2 + \pi^{-2}\,M^{-2},$$

AI.8.12 $$x^2\,M^2 = N'^2 + N^2\,\phi'^2 = N'^2 + \pi^{-2}x^2\,N^{-2},$$

AI.8.13 $$\tan(\theta-\phi) = 1/(\pi\,M\,M') = -\,M\,\theta'\,/\,M',$$

AI.8.14 $$M'' = x\,M + \pi^{-2}\,M^{-3}, \qquad (M^2)''' - 4x(M^2)' - 2\,M^2 = 0$$

AI.8.15 $$\theta'^2 + \tfrac{1}{2}(\theta'''\,/\,\theta') - \tfrac{3}{4}(\theta''\,/\,\theta')^2 = -x.$$

AI.8(iii) Monotonicity

As x increases from $-\infty$ to 0 each of the functions $M(x)$, $M'(x)$, $|x|^{-1/4}\,N(x)$, $M(x)\,N(x)$, $\theta'(x)$, $\phi'(x)$ is increasing, and each of the functions $|x|^{1/4}\,M(x)$, $\theta(x)$, $\phi(x)$ is decreasing.

AI.8(iv) Asymptotic Expansions

As $x \to -\infty$

AI.8.16 $$M^2(x) \sim \frac{1}{\pi(-x)^{1/2}}\sum_{s=0}^{\infty}\frac{1\cdot3\cdot5\cdots(6s-1)}{s!(96)^s}\frac{1}{x^{3s}},$$

AI.8.17 $$N^2(x) \sim \frac{(-x)^{1/2}}{\pi}\sum_{s=0}^{\infty}\frac{1\cdot3\cdot5\cdots(6s-1)}{s!(96)^s}\frac{1+6s}{1-6s}\frac{1}{x^{3s}},$$

AI.8.18 $$\theta(x) \sim \frac{\pi}{4} + \tfrac{2}{3}(-x)^{3/2}\left(1 + \tfrac{5}{32}\frac{1}{x^3} + \tfrac{1105}{6144}\frac{1}{x^6} + \tfrac{82825}{65536}\frac{1}{x^9} + \tfrac{12820\,31525}{587\,20256}\frac{1}{x^{12}} + \cdots\right),$$

AI.8.19 $$\phi(x) \sim -\frac{\pi}{4} + \tfrac{2}{3}(-x)^{3/2}\left(1 - \tfrac{7}{32}\frac{1}{x^3} - \tfrac{1463}{6144}\frac{1}{x^6} - \tfrac{4\,95271}{3\,27680}\frac{1}{x^9} - \tfrac{2065\,30429}{83\,88608}\frac{1}{x^{12}} - \cdots\right).$$

For approximate values (25 significant figures) of the coefficients of the powers x^{-15}, x^{-18}, …, x^{-54} in (AI.8.18) and (AI.8.19) see Sherry (1959).

AI.9 Zeros

AI.9(i) Distribution and Notation

On $\mathbb{R}$ $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, $\mathrm{Bi}(x)$, $\mathrm{Bi}'(x)$ each have an infinite number of zeros, all of which are negative. They are denoted by a_s, a'_s, b_s, b'_s, respectively, arranged in ascending order of absolute value for $s = 1, 2, \ldots$. See §AI.3(i) for visualizations.

In $\mathbb{C}$, $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ have no other zeros. $\mathrm{Bi}(z)$ and $\mathrm{Bi}'(z)$ each have an infinite number of zeros in the sectors $\frac{1}{3}\pi < \mathrm{ph}\,z < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi < \mathrm{ph}\,z < -\frac{1}{3}\pi$. They are denoted by β_s, β'_s, respectively, in the former sector, and by $\bar{\beta}_s$, $\bar{\beta}'_s$, in the conjugate sector, again arranged in ascending order of modulus for $s = 1, 2, \ldots$. See §AI.3(ii) for visualizations.

For the distribution in $\mathbb{C}$ of the zeros of $\mathrm{Ai}'(z) - \sigma\,\mathrm{Ai}(z)$, where σ is an arbitrary complex constant, see Muraveĭ (1976).

AI.9(ii) Relation to Modulus and Phase

AI.9.1 $$\theta(a_s) = \phi(a'_{s+1}) = s\pi,$$

AI.9.2 $$\theta(b_s) = \phi(b'_s) = (s - \tfrac{1}{2})\pi.$$

AI.9.3 $$\mathrm{Ai}'(a_s) = \frac{(-1)^{s-1}}{\pi\,M(a_s)},$$

AI.9.4 $$\mathrm{Bi}'(b_s) = \frac{(-1)^{s-1}}{\pi\,M(b_s)},$$

AI.9.5 $$\mathrm{Ai}(a'_s) = \frac{(-1)^{s-1}}{\pi\,N(a'_s)},$$

AI.9.6 $$\mathrm{Bi}(b'_s) = \frac{(-1)^{s}}{\pi\,N(b'_s)}.$$

AI.9(iii) Asymptotic Expansions

For large s

AI.9.7 $$a_s = -T\left(\tfrac{3}{8}\pi(4s-1)\right),$$

AI.9.8 $$\mathrm{Ai}'(a_s) = (-)^{s-1}V\left(\tfrac{3}{8}\pi(4s-1)\right),$$

AI.9.9 $$a'_s = -U\left(\tfrac{3}{8}\pi(4s-3)\right),$$

AI.9.10 $$\mathrm{Ai}(a'_s) = (-)^{s-1}W\left(\tfrac{3}{8}\pi(4s-3)\right),$$

AI.9.11 $$b_s = -T\left(\tfrac{3}{8}\pi(4s-3)\right),$$

AI.9.12 $$\mathrm{Bi}'(b_s) = (-)^{s-1}V\left(\tfrac{3}{8}\pi(4s-3)\right),$$

AI.9.13 $$b'_s = -U\left(\tfrac{3}{8}\pi(4s-1)\right),$$

AI.9.14 $$\mathrm{Bi}(b'_s) = (-)^{s}W\left(\tfrac{3}{8}\pi(4s-1)\right),$$

AI.9.15 $$\beta_s = e^{\pi i/3}T\left(\tfrac{3}{8}\pi(4s-1) + \tfrac{3}{4}i\ln 2\right),$$

AI.9.16 $$\mathrm{Bi}'(\beta_s) = (-)^{s}\sqrt{2}e^{-\pi i/6}V\left(\tfrac{3}{8}\pi(4s-1) + \tfrac{3}{4}i\ln 2\right),$$

AI.9.17 $$\beta'_s = e^{\pi i/3}U\left(\tfrac{3}{8}\pi(4s-3) + \tfrac{3}{4}i\ln 2\right),$$

AI.9.18 $$\mathrm{Bi}(\beta'_s) = (-)^{s-1}\sqrt{2}e^{\pi i/6}W\left(\tfrac{3}{8}\pi(4s-3) + \tfrac{3}{4}i\ln 2\right),$$

where

AI.9.19 $$T(t) \sim t^{2/3}\left(1 + \tfrac{5}{48}t^{-2} - \tfrac{5}{36}t^{-4} + \tfrac{77125}{82944}t^{-6} - \tfrac{1080\ 56875}{69\ 67296}t^{-8} + \tfrac{16\ 23755\ 96875}{3344\ 30208}t^{-10} - \cdots\right),$$

AI.9.20 $$U(t) \sim t^{2/3}\left(1 - \tfrac{7}{48}t^{-2} + \tfrac{35}{288}t^{-4} - \tfrac{1\ 81223}{2\ 07360}t^{-6} + \tfrac{186\ 83371}{12\ 44160}t^{-8} - \tfrac{9\ 11458\ 84361}{1911\ 02976}t^{-10} + \cdots\right),$$

AI.9.21 $$V(t) \sim \pi^{-1/2}t^{1/6}\left(1 + \tfrac{5}{48}t^{-2} - \tfrac{1525}{4608}t^{-4} + \tfrac{23\ 97875}{6\ 63552}t^{-6} - \tfrac{7\ 48989\ 40625}{8918\ 13888}t^{-8} + \tfrac{14419\ 83037\ 34375}{4\ 28070\ 66024}t^{-10} - \cdots\right),$$

AI.9.22 $$W(t) \sim \pi^{-1/2}t^{-1/6}\left(1 - \tfrac{7}{96}t^{-2} + \tfrac{1673}{6144}t^{-4} - \tfrac{843\ 94709}{265\ 42080}t^{-6} + \tfrac{78\ 02771\ 35421}{1\ 01921\ 58720}t^{-8} - \tfrac{20444\ 90510\ 51945}{6\ 52298\ 15808}t^{-10} + \cdots\right).$$

Error Bounds In the asymptotic expansions for a_s and b_s, the nth error term is bounded in magnitude by the first neglected term, and has the same sign, provided that $n = 1, 2, 3, 4, 5$. The 6th error term has opposite sign to the 5th error term.

In the asymptotic expansions for a'_s and b'_s, the first term provides a lower bound.

AI.9(iv) Tables

Tables AI.9.1–AI.9.4 show real and complex zeros of Ai, Ai′, Bi and Bi′.

Table AI.9.1: Zeros of Ai and Ai′

s	a_s	$\mathrm{Ai}'(a_s)$	a'_s	$\mathrm{Ai}(a'_s)$
1	-2.33810 74105	0.70121 08227	-1.01879 29716	0.53565 66560
2	-4.08794 94441	-0.80311 13697	-3.24819 75822	-0.41901 54780
3	-5.52055 98281	0.86520 40259	-4.82009 92112	0.38040 64686
4	-6.78670 80901	-0.91085 07370	-6.16330 73556	-0.35790 79437
5	-7.94413 35871	0.94733 57094	-7.37217 72550	0.34230 12444
6	-9.02265 08533	-0.97792 28086	-8.48848 67340	-0.33047 62291
7	-10.04017 43416	1.00437 01227	-9.53544 90524	0.32102 22882
8	-11.00852 43037	-1.02773 86888	-10.52766 03970	-0.31318 53910
9	-11.93601 55632	1.04872 06486	-11.47505 66335	0.30651 72939
10	-12.82877 67529	-1.06779 38592	-12.38478 83718	-0.30073 08293

Table AI.9.2: Zeros of Bi and Bi′

s	b_s	$\mathrm{Bi}'(b_s)$	b'_s	$\mathrm{Bi}(b'_s)$
1	-1.17371 32227	0.60195 78880	-2.29443 96826	-0.45494 43836
2	-3.27109 33028	-0.76031 01415	-4.07315 50891	0.39652 28361
3	-4.83073 78417	0.83699 10126	-5.51239 57297	-0.36796 91615
4	-6.16985 21283	-0.88947 99014	-6.78129 44460	0.34949 91168
5	-7.37676 20794	0.92998 36386	-7.94017 86892	-0.33602 62401
6	-8.49194 88465	-0.96323 44302	-9.01958 33588	0.32550 97364
7	-9.53819 43793	0.99158 63705	-10.03769 63349	-0.31693 46537
8	-10.52991 35067	-1.01638 96592	-11.00646 26677	0.30972 59408
9	-11.47695 35513	1.03849 42860	-11.93426 16450	-0.30352 76648
10	-12.38641 71386	-1.05847 18444	-12.82725 83092	0.29810 49111

Table AI.9.3: Complex zeros of Bi

	$e^{-\pi i/3}\beta_s$		$\mathrm{Bi}'(\beta_s)$	
s	modulus	phase	modulus	phase
1	2.35387 33809	0.09533 49591	0.99310 68457	2.64060 02521
2	4.09328 73094	0.04178 55604	1.13612 83345	-0.51328 28720
3	5.52350 35011	0.02668 05442	1.22374 37881	2.62462 83591
4	6.78865 95301	0.01958 69751	1.28822 92493	-0.51871 63829
5	7.94555 90160	0.01547 08228	1.33979 47726	2.62185 44560
6	9.02375 63663	0.01278 34808	1.38303 39005	-0.52040 69437
7	10.04106 73680	0.01089 12610	1.42042 53456	2.62071 41895
8	11.00926 72579	0.00948 68445	1.45346 64633	-0.52122 87219
9	11.93664 76131	0.00840 31785	1.48313 45656	2.62009 35195
10	12.82932 39388	0.00754 16607	1.51010 46383	-0.52171 41947

Table AI.9.4: Complex zeros of Bi′

	$e^{-\pi i/3}\beta'_s$		$\mathrm{Bi}(\beta'_s)$	
s	modulus	phase	modulus	phase
1	1.12139 32942	0.33072 66208	0.75004 14897	0.46597 78930
2	3.25690 82266	0.05938 99367	0.59221 66315	-2.63235 40329
3	4.82400 26102	0.03278 56423	0.53787 06321	0.51549 32992
4	6.16568 66408	0.02266 24588	0.50611 02160	-2.62362 85920
5	7.37383 79870	0.01731 96481	0.48406 00643	0.51928 28169
6	8.48973 85596	0.01401 65283	0.46734 68449	-2.62149 05716
7	9.53644 07072	0.01177 19311	0.45398 23240	0.52066 02139
8	10.52847 37502	0.01014 71783	0.44290 25018	-2.62052 78353
9	11.47574 11237	0.00891 66153	0.43347 44668	0.52137 15495
10	12.38537 59341	0.00795 22843	0.42529 25837	-2.61998 05803

AI.10 Integrals

AI.10.1

$$\int_0^\infty \mathrm{Ai}(t)\,dt = \tfrac{1}{3},$$
$$\int_{-\infty}^0 \mathrm{Ai}(t)\,dt = \tfrac{2}{3}, \quad \int_{-\infty}^0 \mathrm{Bi}(t)\,dt = 0.$$

AI.10(i) Relations to Scorer Functions

AI.10.2

$$\int_z^\infty \mathrm{Ai}(t)\,dt = \pi\left(\mathrm{Ai}(z)\,\mathrm{Gi}'(z) - \mathrm{Ai}'(z)\,\mathrm{Gi}(z)\right),$$

AI.10.3

$$\int_{-\infty}^z \mathrm{Ai}(t)\,dt = \pi\left(\mathrm{Ai}(z)\,\mathrm{Hi}'(z) - \mathrm{Ai}'(z)\,\mathrm{Hi}(z)\right),$$

AI.10.4

$$\begin{aligned}\int_{-\infty}^z \mathrm{Bi}(t)\,dt &= \int_0^z \mathrm{Bi}(t)\,dt \\ &= \pi\left(\mathrm{Bi}'(z)\,\mathrm{Gi}(z) - \mathrm{Bi}(z)\,\mathrm{Gi}'(z)\right) \\ &= \pi\left(\mathrm{Bi}(z)\,\mathrm{Hi}'(z) - \mathrm{Bi}'(z)\,\mathrm{Hi}(z)\right).\end{aligned}$$

For Gi and Hi see §AI.12.

AI.10(ii) Asymptotic Approximations

AI.10.5
$$\int_x^\infty \mathrm{Ai}(t)\,dt \sim \tfrac{1}{2}\pi^{-1/2}x^{-3/4}\exp\left(-\tfrac{2}{3}x^{3/2}\right),\qquad x\to\infty,$$

AI.10.6
$$\int_0^x \mathrm{Bi}(t)\,dt \sim \pi^{-1/2}x^{-3/4}\exp\left(\tfrac{2}{3}x^{3/2}\right),\qquad x\to\infty,$$

AI.10.7
$$\int_{-\infty}^x \mathrm{Ai}(t)\,dt = \pi^{-1/2}(-x)^{-3/4}\cos\left(\tfrac{2}{3}(-x)^{3/2}+\tfrac{1}{4}\pi\right) + O(|x|^{-9/4}),\qquad x\to-\infty,$$

AI.10.8
$$\int_{-\infty}^x \mathrm{Bi}(t)\,dt = \pi^{-1/2}(-x)^{-3/4}\sin\left(\tfrac{2}{3}(-x)^{3/2}+\tfrac{1}{4}\pi\right) + O(|x|^{-9/4}),\qquad x\to-\infty.$$

In (AI.10.5)–(AI.10.8), for additional terms and complex variables, substitute into (AI.10.2)–(AI.10.4) by means of the asymptotic expansions supplied in §AI.7 and AI.12.

AI.10(iii) Other Indefinite Integrals

Let $w(z)$ be any solution of Airy's equation (AI.2.1). Then

AI.10.9
$$\int zw(z)\,dz = w'(z),$$

AI.10.10
$$\int z^2w(z)\,dz = zw'(z)-w(z),$$

AI.10.11
$$\int z^{n+3}w(z)\,dz = z^{n+2}w'(z)-(n+2)z^{n+1}w(z) + (n+1)(n+2)\int z^n w(z)\,dz,$$

where $n=0,1,2,\ldots$.

AI.10(iv) Laplace Transforms

AI.10.12
$$\int_{-\infty}^\infty e^{pt}\,\mathrm{Ai}(t)\,dt = e^{p^3/3},\qquad \Re p>0,$$

AI.10.13
$$\int_0^\infty e^{-pt}\,\mathrm{Ai}(t)\,dt = e^{-p^3/3}\left(\frac{1}{3}-\frac{p\;{}_1F_1(\frac{1}{3};\frac{4}{3};\frac{1}{3}p^3)}{3^{4/3}\Gamma(\frac{4}{3})} + \frac{p^2\;{}_1F_1(\frac{2}{3};\frac{5}{3};\frac{1}{3}p^3)}{3^{5/3}\Gamma(\frac{5}{3})}\right),\qquad p\in\mathbb{C},$$

AI.10.14
$$\int_0^\infty e^{-pt}\,\mathrm{Ai}(-t)\,dt = \frac{1}{3}e^{p^3/3}\left(\frac{\Gamma(\frac{1}{3},\frac{1}{3}p^3)}{\Gamma(\frac{1}{3})}+\frac{\Gamma(\frac{2}{3},\frac{1}{3}p^3)}{\Gamma(\frac{2}{3})}\right),\qquad \Re p>0,$$

AI.10.15
$$\int_0^\infty e^{-pt}\,\mathrm{Bi}(-t)\,dt = \frac{1}{\sqrt{3}}e^{p^3/3}\left(\frac{\Gamma(\frac{2}{3},\frac{1}{3}p^3)}{\Gamma(\frac{2}{3})}-\frac{\Gamma(\frac{1}{3},\frac{1}{3}p^3)}{\Gamma(\frac{1}{3})}\right),\qquad \Re p>0.$$

For the confluent hypergeometric function ${}_pF_q$ and the incomplete Gamma function Γ see Chapters CH and IG.

AI.10(v) Mellin Transform

AI.10.16
$$\int_0^\infty t^{\alpha-1}\,\mathrm{Ai}(t)\,dt = \frac{\Gamma(\alpha)}{3^{(\alpha+2)/3}\Gamma(\frac{1}{3}\alpha+\frac{2}{3})},\qquad \Re\alpha>0.$$

AI.10(vi) Stieltjes Transforms

AI.10.17
$$\mathrm{Ai}(z) = \frac{z^{5/4}e^{-(2/3)z^{3/2}}}{2^{7/2}\pi}\times\int_0^\infty \frac{t^{-1/2}e^{-(2/3)t^{3/2}}\,\mathrm{Ai}(t)}{z^{3/2}+t^{3/2}}\,dt,\qquad |\operatorname{ph} z|<\tfrac{2}{3}\pi,$$

AI.10.18
$$\mathrm{Bi}(x) = \frac{x^{5/4}e^{(2/3)x^{3/2}}}{2^{5/2}\pi}\times P\int_0^\infty \frac{t^{-1/2}e^{-(2/3)t^{3/2}}\,\mathrm{Ai}(t)}{x^{3/2}-t^{3/2}}\,dt,\qquad x>0.$$

P denotes Cauchy principal value.

AI.10(vii) Repeated Integrals

AI.10.19
$$\int_0^x\int_0^v \mathrm{Ai}(t)\,dt\,dv = x\int_0^x \mathrm{Ai}(t)\,dt - \mathrm{Ai}'(x)+\mathrm{Ai}'(0),$$

AI.10.20
$$\int_0^x\int_0^v \mathrm{Bi}(t)\,dt\,dv = x\int_0^x \mathrm{Bi}(t)\,dt - \mathrm{Bi}'(x)+\mathrm{Bi}'(0),$$

AI.10.21
$$\int_0^\infty\int_t^\infty\cdots\int_t^\infty \mathrm{Ai}(-t)(\,dt)^n = \frac{2\cos\left(\frac{1}{3}(n-1)\pi\right)}{3^{(n+2)/3}\Gamma(\frac{1}{3}n+\frac{2}{3})},\qquad n=1,2,\ldots.$$

AI.11 Products

AI.11(i) Differential Equation

AI.11.1
$$\frac{d^3w}{dz^3}-4z\frac{dw}{dz}-2w=0.$$

$w=w_1w_2$, where w_1 and w_2 are any solutions of Airy's equation (AI.2.1). For example, $w=\mathrm{Ai}^2(z)$, $\mathrm{Ai}(z)\,\mathrm{Bi}(z)$, $\mathrm{Ai}(z)\,\mathrm{Ai}(ze^{\mp 2\pi i/3})$, $M^2(z)$. Numerically satisfactory triads of solutions can be constructed where needed on $\mathbb{R}$ or $\mathbb{C}$ by inspection of the asymptotic expansions supplied in §AI.7.

AI.11(ii) Wronskian

AI.11.2 $$\mathscr{W}\left\{\mathrm{Ai}^2(z), \mathrm{Ai}(z)\,\mathrm{Bi}(z), \mathrm{Bi}^2(z)\right\} = 2\pi^{-3}.$$

AI.11(iii) Integral Representations

AI.11.3 $$\mathrm{Ai}^2(x) = \frac{1}{4\pi\sqrt{3}}\int_0^\infty J_0(\tfrac{1}{12}t^3 + xt)\,t\,dt, \qquad x \geq 0.$$

For the Bessel function J_ν, see Chapter BS.

AI.11.4 $$\mathrm{Ai}^2(z) + \mathrm{Bi}^2(z) = \frac{1}{\pi^{3/2}}\int_0^\infty \exp(zt - \tfrac{1}{12}t^3)t^{-1/2}\,dt.$$

AI.11(iv) Indefinite Integrals

Let w_1, w_2 be any solutions of (AI.2.1), not necessarily distinct. Then

AI.11.5 $$\int w_1 w_2\,dz = -w_1' w_2' + z w_1 w_2,$$

AI.11.6 $$\int w_1 w_2'\,dz = \tfrac{1}{2}\left(w_1 w_2 + z\mathscr{W}\{w_1, w_2\}\right),$$

AI.11.7 $$\int w_1' w_2'\,dz = \tfrac{1}{3}(w_1 w_2' + w_1' w_2 + z w_1' w_2' - z^2 w_1 w_2),$$

AI.11.8 $$\int z w_1 w_2\,dz = \tfrac{1}{6}(w_1 w_2' + w_1' w_2) - \tfrac{1}{3}(z w_1' w_2' - z^2 w_1 w_2),$$

AI.11.9 $$\int z w_1 w_2'\,dz = \tfrac{1}{4}\left(w_1' w_2' + z^2\mathscr{W}\{w_1, w_2\}\right),$$

AI.11.10 $$\int z w_1' w_2'\,dz = \tfrac{3}{10}(-w_1 w_2 + z w_1 w_2' + z w_1' w_2) + \tfrac{1}{5}(z^2 w_1' w_2' - z^3 w_1 w_2).$$

For $\int z^n w_1 w_2\,dz$, $\int z^n w_1 w_2'\,dz$, $\int z^n w_1' w_2'\,dz$, where n is any positive integer, see Albright (1977). For related integrals see Gordon (1969, Appendix B).

For any continuously differentiable function f

AI.11.11 $$\int \frac{1}{w_1^2} f'\left(\frac{w_2}{w_1}\right)dz = \frac{1}{\mathscr{W}\{w_1, w_2\}} f\left(\frac{w_2}{w_1}\right).$$

Examples:

AI.11.12 $$\int \frac{dz}{\mathrm{Ai}^2(z)} = \pi\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)}, \qquad \int \frac{dz}{\mathrm{Ai}(z)\,\mathrm{Bi}(z)} = \pi\log\left(\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)}\right),$$

AI.11.13 $$\int \frac{\mathrm{Ai}(z)\,\mathrm{Bi}(z)}{\left(\mathrm{Ai}^2(z) + \mathrm{Bi}^2(z)\right)^2}\,dz = -\frac{\pi}{2}\frac{\mathrm{Ai}^2(z)}{\mathrm{Ai}^2(z) + \mathrm{Bi}^2(z)} = \frac{\pi}{2}\frac{\mathrm{Bi}^2(z)}{\mathrm{Ai}^2(z) + \mathrm{Bi}^2(z)}.$$

AI.12 Scorer Functions

AI.12(i) Differential Equation

AI.12.1 $$\frac{d^2 w}{dz^2} - zw = \frac{1}{\pi}.$$

General solution is given by

AI.12.2 $$w(z) = A w_1(z) + B w_2(z) + p(z),$$

where A and B are arbitrary constants, $w_1(z)$ and $w_2(z)$ are any two linearly independent solutions of Airy's equation (AI.2.1), and $p(z)$ is any particular solution of (AI.12.1). Standard particular solutions are

AI.12.3 $$-\mathrm{Gi}(z),\quad \mathrm{Hi}(z),\quad e^{-2\pi i/3}\,\mathrm{Hi}(ze^{-2\pi i/3}),\quad e^{2\pi i/3}\,\mathrm{Hi}(ze^{2\pi i/3}),$$

with initial values given by

AI.12.4 $$\mathrm{Gi}(0) = \tfrac{1}{2}\,\mathrm{Hi}(0) = \tfrac{1}{3}\,\mathrm{Bi}(0) = 1/\left(3^{7/6}\Gamma(\tfrac{2}{3})\right) = 0.20497\,55424,$$

AI.12.5 $$\mathrm{Gi}'(0) = \tfrac{1}{2}\,\mathrm{Hi}'(0) = \tfrac{1}{3}\,\mathrm{Bi}'(0) = 1/\left(3^{5/6}\Gamma(\tfrac{1}{3})\right) = 0.14942\,94525.$$

AI.12(ii) Numerically Satisfactory Solutions

$-\mathrm{Gi}(x)$ is a numerically satisfactory companion to the complementary functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ on the interval $0 \leq x < \infty$. $\mathrm{Hi}(x)$ is a numerically satisfactory companion to $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ on the interval $-\infty < x \leq 0$.

In $\mathbb{C}$, numerically satisfactory sets of solutions are given by

AI.12.6 $$\mathrm{Gi}(z),\quad \mathrm{Ai}(z),\quad \mathrm{Bi}(z),\quad |\mathrm{ph}\,z| \leq \tfrac{1}{3}\pi,$$

AI.12.7 $$\mathrm{Hi}(z),\quad \mathrm{Ai}(ze^{-2\pi i/3}),\quad \mathrm{Ai}(ze^{2\pi i/3}),\quad |\mathrm{ph}(-z)| \leq \tfrac{2}{3}\pi,$$

AI.12.8 $$e^{-2\pi i/3}\,\mathrm{Hi}(ze^{-2\pi i/3}),\quad \mathrm{Ai}(z),\quad \mathrm{Ai}(ze^{2\pi i/3}),\quad -\pi \leq \mathrm{ph}\,z \leq \tfrac{1}{3}\pi,$$

AI.12.9 $$e^{2\pi i/3}\,\mathrm{Hi}(ze^{2\pi i/3}),\quad \mathrm{Ai}(z),\quad \mathrm{Ai}(ze^{-2\pi i/3}),\quad -\tfrac{1}{3}\pi \leq \mathrm{ph}\,z \leq \pi.$$

AI.12(iii) Connection Formulas

AI.12.10 $$\mathrm{Gi}(z) + \mathrm{Hi}(z) = \mathrm{Bi}(z),$$

AI.12.11 $$\mathrm{Gi}(z) = \tfrac{1}{2}e^{\pi i/3}\,\mathrm{Hi}(ze^{-2\pi i/3}) + \tfrac{1}{2}e^{-\pi i/3}\,\mathrm{Hi}(ze^{2\pi i/3}),$$

AI.12.12 $$\mathrm{Hi}(z) = e^{\pm 2\pi i/3}\,\mathrm{Hi}(ze^{\pm 2\pi i/3}) + 2e^{\mp\pi i/6}\,\mathrm{Ai}(ze^{\mp 2\pi i/3}).$$

AI.12(iv) Maclaurin Series

AI.12.13
$$\mathrm{Gi}(z) = \frac{3^{-2/3}}{\pi} \times \sum_{s=0}^{\infty} \cos\left(\frac{2s-1}{3}\pi\right) \Gamma\left(\frac{s+1}{3}\right) \frac{(3^{1/3}z)^s}{s!},$$

AI.12.14
$$\mathrm{Gi}'(z) = \frac{3^{-1/3}}{\pi} \times \sum_{s=0}^{\infty} \cos\left(\frac{2s+1}{3}\pi\right) \Gamma\left(\frac{s+2}{3}\right) \frac{(3^{1/3}z)^s}{s!},$$

AI.12.15
$$\mathrm{Hi}(z) = \frac{3^{-2/3}}{\pi} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+1}{3}\right) \frac{(3^{1/3}z)^s}{s!},$$

AI.12.16
$$\mathrm{Hi}'(z) = \frac{3^{-1/3}}{\pi} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+2}{3}\right) \frac{(3^{1/3}z)^s}{s!}.$$

AI.12(v) Integral Representations

Indefinite Integrals

AI.12.17
$$\mathrm{Gi}(z) = \mathrm{Bi}(z) \int_z^{\infty} \mathrm{Ai}(t)\,dt + \mathrm{Ai}(z) \int_0^z \mathrm{Bi}(t)\,dt,$$

AI.12.18
$$\mathrm{Hi}(z) = \mathrm{Bi}(z) \int_{-\infty}^{z} \mathrm{Ai}(t)\,dt - \mathrm{Ai}(z) \int_{-\infty}^z \mathrm{Bi}(t)\,dt,$$

AI.12.19
$$\mathrm{Gi}'(z) = \mathrm{Bi}'(z) \int_z^{\infty} \mathrm{Ai}(t)\,dt + \mathrm{Ai}'(z) \int_0^z \mathrm{Bi}(t)\,dt,$$

AI.12.20
$$\mathrm{Hi}'(z) = \mathrm{Bi}'(z) \int_{-\infty}^{z} \mathrm{Ai}(t)\,dt - \mathrm{Ai}'(z) \int_{-\infty}^z \mathrm{Bi}(t)\,dt.$$

Definite Integrals

AI.12.21
$$\mathrm{Gi}(x) = \frac{1}{\pi} \int_0^{\infty} \sin(\tfrac{1}{3}t^3 + xt)\,dt,$$

AI.12.22
$$\mathrm{Hi}(z) = \frac{1}{\pi} \int_0^{\infty} \exp(-\tfrac{1}{3}t^3 + zt)\,dt,$$

AI.12.23
$$\mathrm{Gi}(z) = -\frac{1}{\pi} \int_0^{\infty} \exp\left(-\tfrac{1}{3}t^3 - \tfrac{1}{2}zt\right) \times \cos\left(\tfrac{1}{2}\sqrt{3}zt + \tfrac{2}{3}\pi\right)\,dt.$$

If $\zeta = \frac{2}{3}z^{3/2}$ or $\frac{2}{3}x^{3/2}$, and $K_{1/3}$ is the modified Bessel function (Chapter BS), then

AI.12.24
$$\mathrm{Hi}(-z) = \frac{4z^2}{3^{3/2}\pi^2} \int_0^{\infty} \frac{K_{1/3}(t)}{\zeta^2 + t^2}\,dt, \quad |\mathrm{ph}\,z| < \tfrac{1}{3}\pi,$$

AI.12.25
$$\mathrm{Gi}(x) = \frac{4x^2}{3^{3/2}\pi^2} P \int_0^{\infty} \frac{K_{1/3}(t)}{\zeta^2 - t^2}\,dt, \quad x > 0,$$

where P denotes the Cauchy principal value.

Barnes-Type Integral

AI.12.26
$$\mathrm{Hi}(z) = \frac{3^{-2/3}}{2\pi^2 i} \int_{-i\infty}^{i\infty} \Gamma(\tfrac{1}{3} + \tfrac{1}{3}t)\Gamma(-t)(3^{1/3}e^{\pi i}z)^t\,dt,$$

where the contour of integration separates the poles of $\Gamma(\frac{1}{3} + \frac{1}{3}t)$ from those of $\Gamma(-t)$.

AI.12(vi) Asymptotic Expansions

Functions and Derivatives

AI.12.27
$$\mathrm{Gi}(z) \sim \frac{1}{\pi z} \sum_{s=0}^{\infty} \frac{(3s)!}{s!(3z^3)^s}, \quad |\mathrm{ph}\,z| < \tfrac{1}{3}\pi,$$

AI.12.28
$$\mathrm{Gi}'(z) \sim -\frac{1}{\pi z^2} \sum_{s=0}^{\infty} \frac{(3s+1)!}{s!(3z^3)^s}, \quad |\mathrm{ph}\,z| < \tfrac{1}{3}\pi,$$

AI.12.29
$$\mathrm{Hi}(z) \sim -\frac{1}{\pi z} \sum_{s=0}^{\infty} \frac{(3s)!}{s!(3z^3)^s}, \quad |\mathrm{ph}(-z)| < \tfrac{2}{3}\pi,$$

AI.12.30
$$\mathrm{Hi}'(z) \sim \frac{1}{\pi z^2} \sum_{s=0}^{\infty} \frac{(3s+1)!}{s!(3z^3)^s}, \quad |\mathrm{ph}(-z)| < \tfrac{2}{3}\pi.$$

For other phase ranges combine these results with the connection formulas (AI.12.10)–(AI.12.12) and the asymptotic expansions in §AI.7.

Integrals

AI.12.31
$$\int_0^z \mathrm{Gi}(t)\,dt \sim \frac{1}{\pi}\ln z + \frac{2\gamma + \ln 3}{3\pi} - \frac{1}{\pi} \sum_{s=1}^{\infty} \frac{(3s-1)!}{s!(3z^3)^s}, \quad |\mathrm{ph}\,z| < \tfrac{1}{3}\pi,$$

AI.12.32
$$\int_0^z \mathrm{Hi}(-t)\,dt \sim \frac{1}{\pi}\ln z + \frac{2\gamma + \ln 3}{3\pi} + \frac{1}{\pi} \sum_{s=1}^{\infty} (-)^{s-1} \frac{(3s-1)!}{s!(3z^3)^s}, \quad |\mathrm{ph}\,z| < \tfrac{2}{3}\pi,$$

where γ = Euler's constant (Chapter MP).

AI.12(vii) Graphs

This section to be completed.

AI.13 Generalized Airy Functions

AI.13(i) Generalizations from the Differential Equation

Equations of the form

AI.13.1
$$\frac{d^2w}{dz^2} = z^n w, \quad n = \text{positive integer},$$

are used in approximating solutions to differential equations with multiple turning points; compare §AI.16. From Chapter BS, the general solution of (AI.13.1) is given by

AI.13.2
$$w = z^{1/2}\,\mathscr{Z}_p(\zeta),$$

where

AI.13.3
$$p = \frac{1}{n+2}\,, \quad \zeta = \frac{2}{n+2} z^{(n+2)/2} = 2pz^{1/(2p)}\,,$$

and $\mathscr{Z}_p$ is the modified cylinder function of order p (Chapter BS).

Swanson and Headley (1967) define independent solutions $A_n(z)$ and $B_n(z)$ of (AI.13.1) by

AI.13.4
$$A_n(z) = (2p/\pi)\sin(p\pi) z^{1/2} K_p(\zeta)\,, \qquad B_n(z) = (pz)^{1/2}\left(I_{-p}(\zeta) + I_p(\zeta)\right),$$

when z is real and positive, and by analytic continuation elsewhere. (All solutions of (AI.13.1) are entire functions of z.) When $n = 1$, $A_n(z)$ and $B_n(z)$ become $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, respectively.

Properties of $A_n(z)$ and $B_n(z)$ for general n follow from the corresponding properties of the modified Bessel functions. They include

AI.13.5
$$A_n(0) = p^{1/2} B_n(0) = \frac{p^{1-p}}{\Gamma(1-p)}\,, \qquad -A'_n(0) = p^{1/2} B'_n(0) = \frac{p^p}{\Gamma(p)}\,,$$

AI.13.6
$$A_n(-z) = \begin{cases} pz^{1/2}\left(J_{-p}(\zeta) + J_p(\zeta)\right), & n \text{ odd}, \\ p^{1/2} B_n(z), & n \text{ even}, \end{cases}$$

AI.13.7
$$B_n(-z) = \begin{cases} (pz)^{1/2}\left(J_{-p}(\zeta) - J_p(\zeta)\right), & n \text{ odd}, \\ p^{-1/2} A_n(z), & n \text{ even}, \end{cases}$$

AI.13.8
$$\mathscr{W}\{A_n(z), B_n(z)\} = \frac{2}{\pi} p^{1/2}\sin(p\pi),$$

and for large $|z|$

AI.13.9
$$A_n(z) = \left(\frac{p}{\pi}\right)^{1/2}\sin(p\pi) z^{-n/4} e^{-\zeta}\left(1 + O(\zeta^{-1})\right), \qquad |\mathrm{ph}\, z| < 3p\pi,$$

AI.13.10
$$A_n(-z) = \begin{cases} 2\sqrt{p/\pi}\cos(\tfrac{p\pi}{2}) z^{-n/4}\left(\cos(\zeta - \tfrac{\pi}{4}) + e^{|\Im\zeta|} O(\zeta^{-1})\right), & |\mathrm{ph}\, z| < 2p\pi, \ n \text{ odd}, \\ \sqrt{p/\pi} z^{-n/4} e^{\zeta}\left(1 + O(\zeta^{-1})\right), & |\mathrm{ph}\, z| < p\pi, \ n \text{ even}, \end{cases}$$

AI.13.11
$$B_n(z) = \pi^{-1/2} z^{-n/4} e^{\zeta}\left(1 + O(\zeta^{-1})\right), \qquad |\mathrm{ph}\, z| < p\pi,$$

AI.13.12
$$B_n(-z) = \begin{cases} -\dfrac{2}{\sqrt{\pi}}\sin(\tfrac{p\pi}{2}) z^{-n/4}\left(\sin(\zeta - \tfrac{\pi}{4}) + e^{|\Im\zeta|} O(\zeta^{-1})\right), & |\mathrm{ph}\, z| < 2p\pi, \ n \text{ odd}, \\ \dfrac{1}{\pi}\sin(p\pi) z^{-n/4} e^{-\zeta}\left(1 + O(\zeta^{-1})\right), & |\mathrm{ph}\, z| < 3p\pi, \ n \text{ even}. \end{cases}$$

The distribution in $\mathbb{C}$ and asymptotic properties of the zeros of $A_n(z)$, $A'_n(z)$, $B_n(z)$ and $B'_n(z)$ are investigated in Swanson and Headley (1967) and Headley and Barwell (1975).

Olver (1977), Olver (1978) employed a different normalization from that of Swanson and Headley (1967). In place of (AI.13.1) we have

AI.13.13
$$\frac{d^2w}{dt^2} = \tfrac{1}{4} m^2 t^{m-2} w,$$

where $m = 3, 4, 5, \ldots$. For real variables the solutions of (AI.13.13) are denoted by $U_m(t)$, $U_m(-t)$, when m is even, and by $V_m(t)$, $\bar{V}_m(t)$ when m is odd. Their relations to the functions $A_n(z)$ and $B_n(z)$ are given by

AI.13.14
$$m = n + 2 = 1/p\,, \quad t = (\tfrac{1}{2}m)^{-2/m} z = \zeta^{2/m}\,,$$

AI.13.15
$$\sqrt{2\pi}\left(\frac{m}{2}\right)^{(m-1)/m}\csc\left(\frac{\pi}{m}\right) A_n(z) = \begin{cases} U_m(t), & m \text{ even}, \\ V_m(t), & m \text{ odd}, \end{cases}$$

AI.13.16
$$\sqrt{\pi}\left(\frac{m}{2}\right)^{(m-2)/(2m)}\csc\left(\frac{\pi}{m}\right) B_n(z) = \begin{cases} U_m(-t), & m \text{ even}, \\ \bar{V}_m(t), & m \text{ odd}. \end{cases}$$

Properties and graphs of $U_m(t)$, $V_m(t)$, $\bar{V}_m(t)$ are included in Olver (1977) together with properties and graphs of real solutions of the equation

AI.13.17
$$\frac{d^2w}{dt^2} = -\tfrac{1}{4} m^2 t^{m-2} w,, \qquad m \text{ even},$$

which are denoted by $W_m(t)$, $W_m(-t)$.

In $\mathbb{C}$, the solutions of (AI.13.13) used in Olver (1978) are

AI.13.18
$$W = U_m(te^{-2j\pi i/m}), \quad j = 0, \pm 1, \pm 2, \ldots.$$

$U_m(te^{-2j\pi i/m})$ is recessive in the sector $-(2j-1)\pi/m \le \mathrm{ph}\, z \le (2j+1)\pi/m$, and is therefore an essential member of any numerically satisfactory pair of solutions in this region.

Another normalization of (AI.13.17) is due to Smirnov (1960), and is given by

AI.13.19
$$\frac{d^2w}{ds^2} + s^{\alpha} w = 0,$$

where $-2 < \alpha < \infty$ and $0 < s < \infty$. Solutions are $w = U_1(s,\alpha)$, $U_2(s,\alpha)$, where

AI.13.20
$$U_1(s,\alpha) = \frac{1}{(\alpha+2)^{1/(\alpha+2)}} \Gamma\left(\frac{\alpha+1}{\alpha+2}\right) s^{1/2}\, J_{-1/(\alpha+2)}\left(\frac{2}{\alpha+2} s^{(\alpha+2)/2}\right),$$

AI.13.21
$$U_2(s,\alpha) = (\alpha+2)^{1/(\alpha+2)} \Gamma\left(\frac{\alpha+3}{\alpha+2}\right) s^{1/2}\, J_{1/(\alpha+2)}\left(\frac{2}{\alpha+2} s^{(\alpha+2)/2}\right).$$

The relation of these functions to $W_m(t)$, $W_m(-t)$ is as follows:

AI.13.22
$$\alpha = m-2\,, \quad s = (m/2)^{2/m} t\,,$$

AI.13.23
$$U_1(s,\alpha) = \frac{\pi^{1/2}}{2^{(m+2)/(2m)}\Gamma(1/m)} \left(W_m(t) + W_m(-t)\right),$$

AI.13.24
$$U_2(s,\alpha) = \frac{\pi^{1/2} m^{2/m}}{2^{(m+2)/(2m)}\Gamma(-1/m)} \left(W_m(t) - W_m(-t)\right).$$

AI.13(ii) Generalizations from Integral Representations

Reid (1972) and Drazin and Reid (1981, Appendix) introduced the following contour integrals in constructing approximate solutions to the Orr-Sommerfeld equation for fluid flow:

AI.13.25
$$A_k(z,p) = \frac{1}{2\pi i} \int_{\mathscr{L}_k} t^{-p} \exp(zt - \tfrac{1}{3}t^3)\, dt, \qquad k = 1,2,3, \quad p \in \mathbb{C},$$

AI.13.26
$$B_0(z,p) = \frac{1}{2\pi i} \int_{\mathscr{L}_0} t^{-p} \exp(zt - \tfrac{1}{3}t^3)\, dt, \qquad p = 0, \pm 1, \pm 2, \ldots,$$

AI.13.27
$$B_k(z,p) = \int_{\mathscr{I}_k} t^{-p} \exp(zt - \tfrac{1}{3}t^3)\, dt, \qquad k = 1,2,3, \quad p = 0, \pm 1, \pm 2, \ldots,$$

with $z \in \mathbb{C}$ in all cases. The integration paths $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ are depicted in Figure AI.13.1 and $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$ are depicted in Figure AI.13.2. When p is not an integer the branch of t^{-p} in (AI.13.25) is chosen to be $\exp\left(-p(\ln|t| + i\,\mathrm{ph}\,t)\right)$ with $0 \le \mathrm{ph}\,t < 2\pi$.

When $p = 0$

AI.13.28
$$A_1(z,0) = \mathrm{Ai}(z),$$

AI.13.29
$$A_2(z,0) = e^{2\pi i/3}\,\mathrm{Ai}(ze^{2\pi i/3})\,, \qquad A_3(z,0) = e^{-2\pi i/3}\,\mathrm{Ai}(ze^{-2\pi i/3})\,,$$

AI.13.30
$$B_0(z,0) = 0\,, \quad B_1(z,0) = \pi\,\mathrm{Hi}(z)\,.$$

Each of the functions $A_k(z,p)$ and $B_k(z,p)$ satisfies the differential equation

AI.13.31
$$\frac{d^3w}{dz^3} - z\frac{dw}{dz} + (p-1)w = 0,$$

and the difference equation

AI.13.32
$$f(p-3) - zf(p-1) + (p-1)f(p) = 0.$$

The functions $A_k(z,p)$ are related by

AI.13.33
$$A_2(z,p) = e^{-2(p-1)\pi i/3}\, A_1(ze^{2\pi i/3}, p)\,, \qquad A_3(z,p) = e^{2(p-1)\pi i/3}\, A_1(ze^{-2\pi i/3}, p)\,.$$

Connection formulas for the solutions of (AI.13.31) include

AI.13.34
$$A_1(z,p) + A_2(z,p) + A_3(z,p) + B_0(z,p) = 0,$$

AI.13.35
$$B_2(z,p) - B_3(z,p) = 2\pi i\, A_1(z,p),$$

AI.13.36
$$B_3(z,p) - B_1(z,p) = 2\pi i\, A_2(z,p),$$

AI.13.37
$$B_1(z,p) - B_2(z,p) = 2\pi i\, A_3(z,p).$$

Further properties of these functions, and also of similar contour integrals containing an additional factor $(\ln t)^q$, $q = 1,2,\ldots$, in the integrand, are derived in Reid (1972), Drazin and Reid (1981, Appendix), and Baldwin (1985). These properties include Wronskians, asymptotic expansions, and information on zeros.

AI.14 Incomplete Airy Functions

This section to be completed.

Applications

AI.15 Coalescing Saddle Points

This topic is treated in Chapter IC.

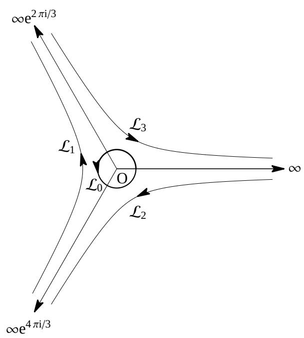


Figure AI.13.1: t-plane paths $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$.

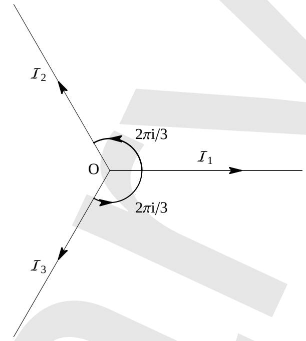


Figure AI.13.2: t-plane paths $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$.

AI.16 Turning-Point Problems

Consider the differential equation

AI.16.1
$$\frac{d^2w}{dx^2} = u^2 f(x) w, \qquad a \le x \le b,$$

in which u is a positive real parameter and the closed interval $[a,b]$ is finite. Suppose that $f(x)$ is real, nonvanishing, and twice-continuously differentiable on $[a,b]$. Then as $u \to \infty$ (AI.16.1) has solutions $w_\pm(u,x)$ such that

AI.16.2
$$w_\pm(u,x) \sim (f(x))^{-1/4} \exp\left(\pm u \int (f(x))^{1/2}\, dx\right),$$

uniformly on $[a,b]$. Moreover, these relations continue to hold when $f(x)$ has a singularity at a or b, or when (a,b) is an infinite open interval, provided that in all cases the "error-control function"

AI.16.3
$$\int |f|^{-1/4} (|f|^{-1/4})''\, dx$$

converges at the endpoint of the interval.

The result just stated is the Liouville-Green (or WKBJ) approximation theorem. The solutions $w_\pm(u,x)$ are exponential or oscillatory in character throughout (a,b), depending whether $f(x)$ is positive or negative.

If (a,b) contains a zero x_0, say, of $f(x)$, then the Liouville-Green approximation breaks down in the neighborhood of x_0. When x_0 is a simple zero, and u is large, the solutions of the differential equation are of exponential type on one side of x_0, and are oscillatory on the other. For this reason x_0 is called a *transition point* or *turning point*, and it is not possible to approximate the solutions in terms of elementary functions uniformly throughout (a,b). But it is possible to construct uniform asymptotic approximations on (a,b) in terms of the solutions of the simplest second-order differential equation with a turning point, namely Airy's equation (AI.2.1).

Without loss of generality assume that the sign of $f(x)/(x-x_0)$ on (a,b) is positive. Assume also that $f(x)/(x-x_0)$ is twice-continuously differentiable on (a,b), and the integral (AI.16.3) converges at each endpoint. Define a new integration variable ζ by

AI.16.4
$$\tfrac{2}{3}(-\zeta)^{3/2} = \int_x^{x_0} (-f(t))^{1/2}\, dt, \quad a < x \le x_0;$$

AI.16.5
$$\tfrac{2}{3}\zeta^{3/2} = \int_{x_0}^{x} (f(t))^{1/2}\, dt, \quad x_0 \le x < b.$$

Then equation (AI.16.1) has solutions $w_1(u,x)$, $w_2(u,x)$ such that

AI.16.6
$$w_1(u,x) = (\zeta/f(x))^{1/4} \left(\mathrm{Bi}(u^{2/3}\zeta) + \epsilon_1(u,x)\right),$$

AI.16.7
$$w_2(u,x) = (\zeta/f(x))^{1/4} \left(\mathrm{Ai}(u^{2/3}\zeta) + \epsilon_2(u,x)\right)$$

where for large u

AI.16.8
$$\epsilon_1(u,x) = M(u^{2/3}\zeta) O(u^{-1}), \qquad \epsilon_2(u,x) = M(u^{2/3}\zeta) O(u^{-1}),$$

uniformly on $(a,x_0]$, and

AI.16.9
$$\epsilon_1(u,x) = \mathrm{Bi}(u^{2/3}\zeta) O(u^{-1}), \qquad \epsilon_2(u,x) = \mathrm{Ai}(u^{2/3}\zeta) O(u^{-1}),$$

uniformly on $[x_0,b)$. Here M is the modulus function introduced in §AI.8.

Full details of this theory, including examples and extensions from asymptotic approximations to asymptotic expansions, complex variables, error bounds, and more general forms of the differential equation than (AI.16.1), may be found in Olver (1974, Chapter 11).

AI.17 Physical Applications

This section to be completed.

Computation

AI.18 Methods of Computation

AI.18(i) Maclaurin Expansions

Although the Maclaurin series expansions of §AI.4 converge for all finite values of z, they are cumbersome to use when $|z|$ is large owing to slowness of convergence and cancellation. For large $|z|$ the asymptotic expansions of §AI.7 should be used instead. Since these expansions diverge, the accuracy they yield is limited by the magnitude of $|z|$. However, this accuracy can be increased considerably by use of the exponentially-improved forms of expansion supplied in §AI.7(iv).

AI.18(ii) Differential Equation

A comprehensive and powerful approach is to integrate the defining differential equation (AI.2.1) by direct numerical methods. As explained in Chapter NM, to insure stability the integration path must be chosen in such a way that as we proceed along it the wanted solution grows at least as fast as all other solutions of the differential equation. In the case of $\mathrm{Ai}(z)$, for example, this means that in the sectors $\frac{1}{3}\pi < |\mathrm{ph}\, z| < \pi$ we may integrate along outward rays from the origin with initial values obtained from (AI.2.3). But when $|\mathrm{ph}\, z| < \frac{1}{3}\pi$ the integration has to be towards the origin, with starting values of $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ computed from their asymptotic expansions. On the remaining rays, given by $\mathrm{ph}\, z = \pm\frac{1}{3}\pi$, π, integration can proceed in either direction.

For details of this method, including a parallelized version, see Lozier and Olver (1993).

AI.18(iii) Integral Representations

Among the integral representations of the Airy functions the Stieltjes transform (AI.10.17) furnishes an effective way of computing $\mathrm{Ai}(z)$ in the complex plane, once values of this function can be generated on the positive real axis. For details, including the application of a generalized form of Gaussian quadrature, see Gordon (1969, Appendix A) and Schulten *et al.* (1979).

AI.18(iv) Via Bessel Functions

Since Airy functions and their derivatives can be expressed as Bessel functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$ (§AI.6(i)), algorithms for generating Bessel functions, including recurrence on the order (Chapter BS), can be used to generate $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$, and their derivatives. Some software packages incorporate this feature; see §AI.21.

AI.18(v) Zeros

Zeros can be computed to high precision by Newton's method (Chapter NM), using values supplied by the asymptotic expansions of §AI.9 as initial approximations. This method was used in the computation of the tables in §AI.9(iv).

AI.18(vi) Scorer Functions

Methods similar to those of §§AI.18(ii), AI.18(iii), and AI.18(iv), can be used for Scorer functions, $\mathrm{Gi}(z)$ and $\mathrm{Hi}(z)$.

Integration of the differential equation (AI.12.1) is more difficult than (AI.2.1), however, because in some regions stable directions of integration do not exist. An example is provided by $\mathrm{Gi}(x)$ on the positive real axis. In these cases boundary-value methods can be used instead; see for example Olde Daalhuis and Olver (1998).

For an application of generalized Gaussian quadrature to the integrals (AI.12.22) and (AI.12.25), see Gordon (1970, Appendix A). Other integral representations can also be used, for example, (AI.12.22), (AI.12.23); see Lee (1980).

AI.19 Tables

The notation 8S, or 8D, signifies 8 significant figures, or decimal digits, respectively.

AI.19(i) Real Variables

- Miller (1946) includes $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$ for $-20 \le x \le 2$; $\log_{10} \mathrm{Ai}(x)$, $\mathrm{Ai}'(x)/\mathrm{Ai}(x)$ for $0 \le x \le 75$; $\mathrm{Bi}(x)$, $\mathrm{Bi}'(x)$ for $-10 \le x \le 2.5$; $\log_{10} \mathrm{Bi}(x)$, $\mathrm{Bi}'(x)/\mathrm{Bi}(x)$ for $0 \le x \le 10$; $M(x)$, $N(x)$, $\theta(x)$, $\phi(x)$ (respectively $F(x)$, $G(x)$, $\chi(x)$, $\psi(x)$) for $-80 \le x \le 0$. Precision is generally 8D; slightly less for some of the auxiliary functions. Extracts from these tables are included in Abramowitz and Stegun (1964, Chapter 10), as well as some auxiliary functions for large arguments.

- Zhang and Jin (1996) tabulate $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, $\mathrm{Bi}(x)$, $\mathrm{Bi}'(x)$ for $0 \le x \le 20$ to 8S and for $-20 \le x \le 0$ to 9D.

- Yakovleva (1969) tabulates Fock's functions (AI.1.2) $U(x) \equiv \sqrt{\pi}\,\mathrm{Bi}(x)$, $U'(x) \equiv \sqrt{\pi}\,\mathrm{Bi}'(x)$, $V(x) = \sqrt{\pi}\,\mathrm{Ai}(x)$, $V'(x) = \sqrt{\pi}\,\mathrm{Ai}'(x)$ for $-9 \le x \le 9$. Precision is 7S.

AI.19(ii) Complex Variables

- Woodward and Woodward (1946) tabulate the real and imaginary parts of $\mathrm{Ai}(z)$, $\mathrm{Ai}'(z)$, $\mathrm{Bi}(z)$, $\mathrm{Bi}'(z)$ for $-2.4 \le \Re z \le 2.4$, $-2.4 \le \Im z \le 0$. Precision is 4D.

- Harvard (1945) tabulates the real and imaginary parts of $h_1(z)$, $h_1'(z)$, $h_2(z)$, $h_2'(z)$ for $-x_0 \le \Re z \le x_0$, $0 \le \Im z \le y_0$, $|x_0 + iy_0| < 6.1$, where

$$h_1(z) = -2^{4/3}3^{1/6} i \operatorname{Ai}(e^{-\pi i/3} z)\,,$$
$$h_2(z) = 2^{4/3}3^{1/6} i \operatorname{Ai}(e^{\pi i/3} z)\,.$$

Precision is 8D.

AI.19(iii) Zeros

- Miller (1946) includes a_s, $\operatorname{Ai}'(a_s)$, a_s', $\operatorname{Ai}(a_s')$ for $1 \le s \le 50$; b_s, $\operatorname{Bi}'(b_s)$, b_s', $\operatorname{Bi}(b_s')$ for $1 \le s \le 20$. Precision is 8D. Entries for $1 \le s \le 10$ are reproduced in Abramowitz and Stegun (1964, Chapter 10).

- Sherry (1959) gives a_s, $\operatorname{Ai}'(a_s)$, a_s', $\operatorname{Ai}(a_s')$ for $1 \le s \le 50$ to 20S.

- Zhang and Jin (1996) include a_s, $\operatorname{Ai}'(a_s)$, a_s', $\operatorname{Ai}(a_s')$, b_s, $\operatorname{Bi}'(b_s)$, b_s', $\operatorname{Bi}(b_s')$ for $1 \le s \le 20$ to 8D.

- Olver (1954) includes β_s, $\operatorname{Bi}'(\beta_s)$, β_s', $\operatorname{Bi}(\beta_s')$ in modulus and phase form for $1 \le s \le 5$ to 3D. These values are reproduced in Abramowitz and Stegun (1964, Chapter 10).

- Corless *et al.* (1992) give the real and imaginary parts of β_s for $1 \le s \le 13$ to 14S.

AI.19(iv) Integrals

- Rothman (1954a) tabulates $\int_0^x \operatorname{Ai}(t)\,dt$ and $\int_0^x \operatorname{Bi}(t)\,dt$ for $-10 \le x < \infty$ and $-10 \le x \le 2$, respectively, to 7D. The entries in the columns headed $\int_0^x \operatorname{Ai}(-x)\,dx$ and $\int_0^x \operatorname{Bi}(-x)\,dx$ all have the wrong sign. The tables are reproduced in Abramowitz and Stegun (1964, Chapter 10), and the sign errors are corrected in later reprintings.

- NBS (1958) includes $\int_0^x \operatorname{Ai}(-t)\,dt$ and $\int_0^x \int_0^v \operatorname{Ai}(-t)\,dt\,dv$ (see (AI.10.19)) for $-2 \le x \le 5$ to 8D and 7D, respectively.

- Zhang and Jin (1996) include $\int_0^x \operatorname{Ai}(t)\,dt$ and $\int_0^x \operatorname{Bi}(t)\,dt$ for $-10 \le x \le 10$ to 8D or 8S.

AI.19(v) Scorer Functions

- Scorer (1950) tabulates $\operatorname{Gi}(x)$ and $\operatorname{Hi}(-x)$ for $0 \le x \le 10$ to 7D.

- Rothman (1954b) tabulates $\int_0^x \operatorname{Gi}(t)\,dt$, $\operatorname{Gi}'(x)$, $\int_0^x \operatorname{Hi}(-t)\,dt$, $-\operatorname{Hi}'(-x)$ for $0 \le x \le 10$ to 7D.

- NBS (1958) includes $A_0(x) \equiv \pi \operatorname{Hi}(-x)$ and $-A_0'(x) \equiv \pi \operatorname{Hi}'(-x)$ for $0 \le x \le 11$ and $0 < 1/x \le 0.1$; $\int_0^x A_0(t)\,dt$ for $0 \le x \le 11$. Precision is 8D.

- Nosova and Tumarkin (1965) include $e_0(x) \equiv \pi \operatorname{Hi}(-x)$, $e_0'(x) \equiv -\pi \operatorname{Hi}'(-x)$, $\tilde{e}_0(-x) \equiv -\pi \operatorname{Gi}(x)$, $\tilde{e}_0'(-x) \equiv \pi \operatorname{Gi}'(x)$ for $-1 \le x \le 10$ to 7D. Also included are the real and imaginary parts of $e_0(z)$ and $ie_0'(z)$, where $z = iy$ and $0 \le y \le 9$, to 6-7D.

AI.19(vi) Generalized Airy Functions

- Smirnov (1960) includes $U_1(s,\alpha)$, $U_2(s,\alpha)$, defined by (AI.13.20), (AI.13.21), and also $\partial U_1(s,\alpha)/\partial s$, $\partial U_2(s,\alpha)/\partial s$, for $\alpha = 1$, $-6 \le s \le 10$ to 5D or 5S, and $-0.75 \le \alpha \le 2$, $0 \le s \le 6$ to 4D.

AI.20 Approximations

The notation 8S, or 8D, signifies 8 significant figures, or decimal digits, respectively.

AI.20(i) Approximations in Terms of Elementary Functions

- Martín *et al.* (1992) provide two simple formulas for approximating $\operatorname{Ai}(x)$ to graphical accuracy, one for $-\infty < x \le 0$, the other for $0 \le x < \infty$.

- Moshier (1989, §6.14) provides minimax rational approximations for calculating $\operatorname{Ai}(x)$, $\operatorname{Ai}'(x)$, $\operatorname{Bi}(x)$, $\operatorname{Bi}'(x)$. They are in terms of the variable ζ, where $\zeta = \frac{2}{3}x^{3/2}$ when x is positive, or $\zeta = \frac{2}{3}(-x)^{3/2}$ when x is negative. The approximations apply when $2 \le \zeta < \infty$, that is, when $3^{2/3} \le x < \infty$ or $-\infty < x \le -3^{2/3}$. The precision in the coefficients is 21S.

AI.20(ii) Expansions in Chebyshev Series

These expansions are for real arguments x and are supplied in sets of four for each function, corresponding to intervals $-\infty < x \le a$, $a \le x \le 0$, $0 \le x \le b$, $b \le x < \infty$. The constants a and b are chosen numerically, with a view to equalizing the effort required for summing the series in each of the four cases.

- Prince (1975) covers $\operatorname{Ai}(x)$, $\operatorname{Ai}'(x)$, $\operatorname{Bi}(x)$, $\operatorname{Bi}'(x)$. The Chebyshev coefficients are given to 10-11D. Fortran programs are included.

- Németh (1971) covers $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, $\mathrm{Bi}(x)$, $\mathrm{Bi}'(x)$, and integrals $\int_0^x \mathrm{Ai}(t)\,dt$, $\int_0^x \mathrm{Bi}(t)\,dt$, $\int_0^x\int_0^v \mathrm{Ai}(t)\,dt\,dv$, $\int_0^x\int_0^v \mathrm{Bi}(t)\,dt\,dv$ (see also (AI.10.19) and (AI.10.20)). The Chebyshev coefficients are given to 15D. Chebysev coefficients are also given for expansions of the second and higher (real) zeros of $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, $\mathrm{Bi}(x)$, $\mathrm{Bi}'(x)$, again to 15D.

- Razaz and Schonfelder (1980) covers $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, $\mathrm{Bi}(x)$, $\mathrm{Bi}'(x)$. The Chebyshev coefficients are given to 30D.

AI.20(iii) Approximations in the Complex Plane

- Corless *et al.* (1992) describe a method of approximation based on subdividing $\mathbb{C}$ into a triangular mesh, with values of $\mathrm{Ai}(z)$, $\mathrm{Ai}'(z)$ stored at the nodes. $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ are then computed from Taylor-series expansions centered at one of the nearest nodes. The Taylor coefficients are generated by recursion, starting from the stored values of $\mathrm{Ai}(z)$, $\mathrm{Ai}'(z)$ at the node. Similarly for $\mathrm{Bi}(z)$, $\mathrm{Bi}'(z)$.

AI.21 Software

DLMF ⇐ See the Digital Library of Mathematical Functions at `HTTP://dlmf.nist.gov/`.

References

General References

The main references used in writing this chapter are Miller (1946) and Olver (1974). For additional bibliographic reading see Bleistein and Handelsman (1975), Jeffreys and Jeffreys (1966), Lebedev (1965), Temme (1996), Wasow (1965), Wasow (1985) and Wong (1989).

Original Sources

The following list gives the sources used in constructing the various sections this chapter. These sources supplement the references that are quoted in the text.

§AI.2 Miller (1946), Olver (1974, Chapter 11), Smith (1990).

§AI.3 These graphs were computed at NIST.

§AI.4 Miller (1946), Olver (1974, Chapter 2).

§AI.5 Miller (1946), Olver (1974, Chapters 2 and 11).

§AI.6 Miller (1946), Olver (1974, Chapter 11).

§AI.7 Olver (1974, Chapters 7 and 11), Olver (1991), Olver (1993).

§AI.8 Miller (1946), Olver (1974, Chapter 11).

§AI.9 Miller (1946), Olver (1974, Chapter 11), Olver (1954), Fabijonas and Olver (1999), Pittaluga and Sacripante (1991). The numerical tables were computed at NIST.

§AI.10 Olver (1974, Chapters 9 and 11), Gordon (1970, Appendix B), Widder (1979), Gibbs (1973, problem 72-21), Schulten *et al.* (1979).

§AI.11 Miller (1946), Olver (1974, Chapter 2), Lebedev (1965, Chapter 5), Muldoon (1977), Albright (1977), Albright and Gavathas (1986).

§AI.12 Olver (1974, Chapter 11), Lee (1980), Gordon (1970, Appendix A), Exton (1983), Rothman (1954b).

Bibliography

M. Abramowitz and I. A. Stegun (Eds.) (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Number 55 in National Bureau of Standards Applied Mathematics Series. U.S. Government Printing Office, Washington, D.C.

J. R. Albright (1977). *Integrals of products of Airy functions.* J. Phys. A **10**(4), pp. 485–490.

J. R. Albright and E. P. Gavathas (1986). *Integrals involving Airy functions.* J. Phys. A **19**(13), pp. 2663–2665.

D. E. Amos (1986). *Algorithm 644: A portable package for Bessel functions of a complex argument and nonnegative order.* ACM Trans. Math. Software **12**(3), pp. 265–273. For remark see same journal v. 16 (1990), no. 4, p. 404.

P. Baldwin (1985). *Zeros of generalized Airy functions.* Mathematika **32**(1), pp. 104–117.

N. Bleistein and R. A. Handelsman (1975). *Asymptotic Expansions of Integrals.* New York: Holt, Rinehart, and Winston. Reprinted with corrections by Dover Publications Inc., New York, 1986.

R. M. Corless, D. J. Jeffrey, and H. Rasmussen (1992). *Numerical evaluation of Airy functions with complex arguments.* J. Comput. Phys. **99**(1), pp. 106–114.

P. G. Drazin and W. H. Reid (1981). *Hydrodynamic Stability.* Cambridge: Cambridge University Press. Cambridge Monographs on Mechanics and Applied Mathematics.

H. Exton (1983). *The asymptotic behaviour of the inhomogeneous Airy function* $\mathrm{Hi}(z)$. Math. Chronicle **12**, pp. 99–104.

B. R. Fabijonas and F. W. J. Olver (1999). *On the reversion of an asymptotic expansion and the zeros of the Airy functions.* SIAM Rev. **41**(4), pp. 762–773.

V. Fock (1945). *Diffraction of radio waves around the earth's surface.* Acad. Sci. USSR. J. Phys. **9**, pp. 255–266.

A. G. Gibbs (1973). *Problem 72-21, Laplace transforms of Airy functions.* SIAM Rev. **15**(4), pp. 796–798. Problem proposed by P. Smith.

R. G. Gordon (1969). *New method for constructing wavefunctions for bound states and scattering.* J. Chem. Phys. **51**, pp. 14–25.

R. G. Gordon (1970). *Constructing wavefunctions for nonlocal potentials.* J. Chem. Phys. **52**, pp. 6211–6217.

Harvard (1945). Harvard University. *Tables of the Modified Hankel Functions of Order One-Third and of their Derivatives.* Cambridge, MA: Harvard University Press.

V. B. Headley and V. K. Barwell (1975). *On the distribution of the zeros of generalized Airy functions.* Math. Comp. **29**, pp. 863–877.

H. Jeffreys (1928). *The effect on Love waves of heterogeneity in the lower layer.* Monthly Notices Roy. Astronom. Soc. Geophysical Supplement **2**, pp. 101–111.

H. Jeffreys and B. S. Jeffreys (1966). *Methods of Mathematical Physics* (Third ed.). Cambridge: Cambridge University Press. Reprinted in 1999.

N. N. Lebedev (1965). *Special Functions and Their Applications.* Englewood Cliffs, N.J.: Prentice-Hall Inc. R. A. Silverman, translator and editor; reprinted by Dover, New York, 1972.

S.-Y. Lee (1980). *The inhomogeneous Airy functions,* $\mathrm{Gi}(z)$ *and* $\mathrm{Hi}(z)$. J. Chem. Phys. **72**(1), pp. 332–336.

D. W. Lozier and F. W. J. Olver (1993). *Airy and Bessel functions by parallel integration of ODEs.* In R. Sincovec *et al.* (Eds.), *Sixth SIAM Conference on Parallel Processing for Scientific Computing*, pp. 530–538. Philadelphia:SIAM.

P. Martín, R. Pérez, and A. L. Guerrero (1992). *Two-point quasi-fractional approximations to the Airy function* $\mathrm{Ai}(x)$. J. Comput. Phys. **99**(2), pp. 337–340.

J. C. P. Miller (1946). *The Airy integral.* In *British Association for the Advancement of Science, Mathematical Tables* (Part-Vol. B). Cambridge: Cambridge University Press.

S. L. B. Moshier (1989). *Methods and Programs for Mathematical Functions.* Chichester: Ellis Horwood Ltd. Includes diskette.

M. E. Muldoon (1977). *Higher monotonicity properties of certain Sturm-Liouville functions. V.* Proc. Roy. Soc. Edinburgh Sect. A **77**(1–2), pp. 23–37.

L. A. Muraveĭ (1976). *Zeros of the function* $\mathrm{Ai}'(z) - \sigma\,\mathrm{Ai}(z)$. Differential Equations **11**, pp. 797–811.

NBS (1958). National Bureau of Standards. *Integrals of Airy Functions.* Number 52 in National Bureau of Standards Applied Mathematics Series. Washington, D.C.: U.S. Government Printing Office.

G. Németh (1971). *Chebyshev polynomial expansions of Airy functions, their zeros, derivatives, first and second integrals.* Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **20**, pp. 13–33.

L. N. Nosova and S. A. Tumarkin (1965). *Tables of Generalized Airy Functions for the Asymptotic Solution of the Differential Equations* $\epsilon(py')' + (q + \epsilon r)y = f$. Oxford: Pergamon Press. D. E. Brown, translator.

A. B. Olde Daalhuis (1995). *Hyperasymptotic solutions of second-order linear differential equations. II.* Methods Appl. Anal. **2**(2), pp. 198–211.

A. B. Olde Daalhuis (1996). *Hyperterminants. I.* J. Comput. Appl. Math. **76**(1-2), pp. 255–264.

A. B. Olde Daalhuis and F. W. J. Olver (1995). *Hyperasymptotic solutions of second-order linear differential equations. I.* Methods Appl. Anal. **2**(2), pp. 173–197.

A. B. Olde Daalhuis and F. W. J. Olver (1998). *On the asymptotic and numerical solution of linear ordinary differential equations.* SIAM Rev. **40**(3), pp. 463–495.

F. W. J. Olver (1954). *The asymptotic expansion of Bessel functions of large order.* Philos. Trans. Roy. Soc. London. Ser. A. **247**, pp. 328–368.

F. W. J. Olver (1974). *Asymptotics and Special Functions.* New York-London: Academic Press. Reprinted by A. K. Peters, Wellesley, MA, 1997.

F. W. J. Olver (1977). *Connection formulas for second-order differential equations with multiple turning points.* SIAM J. Math. Anal. **8**(1), pp. 127–154.

F. W. J. Olver (1978). *General connection formulae for Liouville-Green approximations in the complex plane.* Philos. Trans. Roy. Soc. London Ser. A **289**, pp. 501–548.

F. W. J. Olver (1991). *Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms.* SIAM J. Math. Anal. **22**(5), pp. 1475–1489.

F. W. J. Olver (1993). *Exponentially-improved asymptotic solutions of ordinary differential equations I: The confluent hypergeometric function.* SIAM J. Math. Anal. **24**(3), pp. 756–767.

G. Pittaluga and L. Sacripante (1991). *Inequalities for the zeros of the Airy functions.* SIAM J. Math. Anal. **22**(1), pp. 260–267.

P. J. Prince (1975). *Algorithm 498: Airy functions using Chebyshev series approximations.* ACM Trans. Math. Software **1**(4), pp. 372–379.

M. Razaz and J. L. Schonfelder (1980). *High precision Chebyshev expansions for Airy functions and their derivatives.* Technical report, University of Birmingham Computer Centre.

W. H. Reid (1972). *Composite approximations to the solutions of the Orr-Sommerfeld equation.* Studies in Appl. Math. **51**, pp. 341–368.

M. Rothman (1954a). *The problem of an infinite plate under an inclined loading, with tables of the integrals of* $\mathrm{Ai}(\pm x)$ *and* $\mathrm{Bi}(\pm x)$. Quart. J. Mech. Appl. Math. **7**, pp. 1–7.

M. Rothman (1954b). *Tables of the integrals and differential coefficients of* $\mathrm{Gi}(+x)$ *and* $\mathrm{Hi}(-x)$. Quart. J. Mech. Appl. Math. **7**, pp. 379–384.

Z. Schulten, D. G. M. Anderson, and R. G. Gordon (1979). *An algorithm for the evaluation of the complex Airy functions.* J. Comput. Phys. **31**(1), pp. 60–75.

R. S. Scorer (1950). *Numerical evaluation of integrals of the form* $I = \int_{x_1}^{x_2} f(x) e^{i\phi(x)}\,dx$ *and the tabulation of the function* $\mathrm{Gi}(z) = (1/\pi)\int_0^\infty \sin(uz + \frac{1}{3}u^3)\,du$. Quart. J. Mech. Appl. Math. **3**, pp. 107–112.

M. E. Sherry (1959). *The zeros and maxima of the Airy function and its first derivative to 25 significant figures.* Report afcrc-tr-59-135, astia document no. ad214568, Air Research and Development Command, U.S. Air Force, Bedford, MA. Available from U.S. Department of Commerce, Office of Technical Services, Washington, D.C.

A. D. Smirnov (1960). *Tables of Airy Functions and Special Confluent Hypergeometric Functions.* Oxford: Pergamon Press. D. G. Fry, translator.

D. R. Smith (1990). *A Riccati approach to the Airy equation.* In R. Wong (Ed.), *Asymptotic and computational analysis (Winnipeg, MB, 1989)*, pp. 403–415. New York: Marcel Dekker.

C. A. Swanson and V. B. Headley (1967). *An extension of Airy's equation.* SIAM J. Appl. Math. **15**, pp. 1400–1412.

N. M. Temme (1996). *Special Functions: An Introduction to the Classical Functions of Mathematical Physics.* New York: John Wiley & Sons Inc.

S. A. Tumarkin (1959). *Asymptotic solution of a linear nonhomogeneous second order differential equation with a transition point and its application to the computations of toroidal shells and propeller blades.* J. Appl. Math. Mech. **23**, pp. 1549–1565.

W. Wasow (1965). *Asymptotic Expansions for Ordinary Differential Equations.* New York-London-Sydney: Interscience Publishers John Wiley & Sons, Inc. Reprinted by Krieger, New York, 1976 and Dover, New York, 1987.

W. Wasow (1985). *Linear Turning Point Theory.* Applied Mathematical Sciences No. 54. New York: Springer-Verlag.

D. V. Widder (1979). *The Airy transform.* Amer. Math. Monthly **86**(4), pp. 271–277.

R. Wong (1989). *Asymptotic Approximations of Integrals.* Boston-New York: Academic Press Inc. Reprinted with corrections by SIAM, Philadelphia, PA, 2001.

P. M. Woodward and A. M. Woodward (1946). *Four-figure tables of the Airy function in the complex plane.* Philos. Mag. (7) **37**, pp. 236–261.

G. D. Yakovleva (1969). *Tables of Airy Functions and their Derivatives.* Moscow: Izdat. Nauka.

S. Zhang and J. Jin (1996). *Computation of Special Functions.* New York: John Wiley & Sons Inc. Includes diskette with Fortran programs.

Index

Metadata